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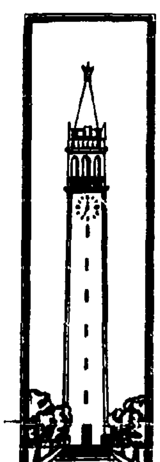
R. P. NORDGREN



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Abstract

The method of Green's function is applied to the quasi-static thermoelastic theory of shallow shells with heat conduction equations included. Solution formulas are derived for middle-surface displacements and stress and temperature resultants in terms of initial and edge temperatures, internal heat sources, ambient temperatures at the upper and lower surfaces, and surface tractions. Equations are given for the Green's functions appearing in the solution formulas. Extension to more general shell theory is discussed. By way of example, the method is applied to thermoelastic problems for two classes of shallow shells. Also, the effect of transverse shear deformation is examined with reference to a shallow spherical shell.

1. INTRODUCTION

This paper applies the method of Green's function to quasi-static thermoelastic problems in the theory of shallow shells. Thermoelastic equations for shallow shells follow from the original work of Marguerre (1938)* upon addition of the effect of thermal expansion. The two temperature resultants appearing in these equations are governed by two-dimensional heat conduction equations derived by Bolotin (1960) for thin shells. For flat plates Bolotin's equations reduce to those of Marguerre (1935)

* The nonlinear terms contained in Marguerre's equations will not be included here.

and for shallow shells they are independent of the form of the shell middle surface.

Employing the indicated equations, solution formulas are derived which express the middle-surface displacements and the stress and temperature resultants in terms of the initial temperature, edge temperature, heat sources within the shell, and ambient temperatures at the upper and lower surfaces. The two sets of Green's functions in these integral solution formulas are shown to satisfy the combined thermoelastic equations with the two resultant heat-supply terms replaced by δ-functions. A related solution formula for shallow shells under surface and edge traction is also obtained, together with equations for the Green's functions appearing in it.

The solution formulas obtained extend immediately to a simplified theory of shells (sometimes referred to as the "technical" theory of shells) presented, e.g., by Green and Zerna (1954). Extension to more general shell theory is possible and is discussed. The method of derivation of the thermoelastic solution formulas is applicable to other special theories of elasticity as well as to the three-dimensional theory. In this connection it should be recalled that Parkus (1959, p. 13) gives a solution formula for stress in quasi-static thermoelastic problems on the basis of analogy and an intuitive argument. Also, integral formulas for several thermoelastic problems with unspecified temperature distribution are derived by Goodier (1958) and Goodier and Nevill (1961). Singular solutions for shallow spherical and cylindrical shells with concentrated temperature resultants are given by Flügge and Conrad (1956, 1958).**

The solution formulas derived here are applied to thermoelastic problems for two classes of shallow shells, namely, unlimited shallow shells with quadratic middle surfaces and rectangular shallow shells with simply supported edges parallel to the principal axis of quadratic middle surfaces. With the aid of a method of spectral representation established by Friedman (1956), the thermoelastic Green's functions are obtained for the former class in the form of a Fourier series of Hankel transform integrals and for the latter class in the form of convergent double Fourier series. Both representations

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**Previously, one of these singular solutions for a shallow cylindrical shell was proposed by Rabotnov (1946) as the solution for a concentrated normal force. However, an algebraic mistake occurred and, as shown by Flügge and Conrad (1958), the concentrated force solution cannot be represented by a singularity of this type.

appear suited to numerical evaluation. For three members of the class of unlimited shallow shells, the Green's function is evaluated at the source point as a function of time in terms of tabulated functions and comparison is made with the dynamic thermoelastic Green's functions for bending of a flat plate. An example of edge heating is studied for the class of rectangular shallow shells.

In the last section, the effect of transverse shear deformation is examined with reference to an unlimited shallow spherical shell (parabaloid of revolution) under specified temperature field and normal surface traction. A quasi-static solution is obtained by the method of Green's function with use of the extended definition of the Laplacian operator following Friedman (1956). The character of the fundamental singularities is compared with that of previous solutions by Reissner (1946) and by Flügge and Conrad (1956) according to classical theory, which neglects the effect of transverse shear deformation.

2. QUASI-STATIC THERMOELASTIC THEORY OF SHELLS

2.1 <u>A Solution Formula for Heat Conduction</u>. The two-dimensional heat conduction equations for thin shells derived recently by Bolotin (1960) from a general variational principle for three-dimensional isotropic (uncoupled) heat conduction with linear heat transfer at the boundary may be written as

$$\begin{aligned} \nabla^2\theta_N - \eta_N\theta_N - \frac{1}{\kappa}\frac{\partial\theta_N}{\partial t} &= -Q_N , \\ \nabla^2\theta_M - \eta_M\theta_M - \frac{1}{\kappa}\frac{\partial\theta_N}{\partial t} &= -Q_M , \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} &\eta_N = \frac{2H}{hK} , \quad \eta_M = \frac{12}{h^2} + \frac{6H}{hK} , \quad \kappa = \frac{K}{\rho c} , \\ &\theta_N = \frac{1}{h}\int_{-\frac{h}{2}}^{+\frac{h}{2}} T dx^3 , \quad \theta_M = \frac{12}{h^3}\int_{-\frac{h}{2}}^{+\frac{h}{2}} Tx^3 dx^3 , \\ &Q_M = \frac{1}{Kh}\int_{-\frac{h}{2}}^{+\frac{h}{2}} Q dx^3 + \frac{H}{hK}\left(\tilde{T}_+ + \tilde{T}_-\right), \\ &Q_N = \frac{12}{Kh^3}\int_{-\frac{h}{2}}^{+\frac{h}{2}} Qx^3 dx^3 + \frac{6H}{h^2K}\left(\tilde{T}_+ + \tilde{T}_-\right). \end{aligned} \tag{2.2}$$

In (2.1) the Laplacian is $\nabla^2 = a^{\alpha\beta}\nabla_\alpha\nabla_\beta$, where the covariant derivative operator ∇_α refers to a curvilinear coordinate system x^α ($\alpha = 1, 2$) on the middle surface of the shell with metric $a_{\alpha\beta}$ and conjugate metric $a^{\alpha\beta}$, the summation convention being understood. The thickness of the shell is denoted by h and x^3 is the coordinate normal to the middle surface. The temperature resultants θ_N and θ_M are expressed in (2.2) as integrals of the temperature T (above a fixed reference temperature), and the derivation by Bolotin (1960) is based on the assumption

$$T = \theta_N(x^\alpha) + x^3\theta_M(x^\alpha). \tag{2.3}$$

The heat inputs Q_M and Q_N arise either from internal heat sources of rate Q per unit time per unit volume or from heat flux across the upper and lower surfaces of the shell where the ambient temperatures are $\tilde{T}_+$ and $\tilde{T}_-$, respectively. The surface conductance H, thermal conductivity K, specific heat c, and mass density ρ are assumed independent of temperature as well as coordinates. On the boundary curve C of the shell middle surface the temperature resultants may be specified, i.e.,

$$\theta_N = \tilde{\theta}_N, \qquad \theta_M = \tilde{\theta}_M, \qquad \text{on C} \tag{2.4}$$

which is a special case of the linear heat transfer edge conditions given by Bolotin (1960). In addition, θ_N and θ_M must satisfy the initial conditions

$$\lim_{t \to 0} \theta_N(x^\alpha, t) = \theta_N^\circ(x^\alpha), \qquad \lim_{t \to 0} \theta_M(x^\alpha, t) = \theta_M^\circ(x^\alpha), \tag{2.5}$$

where $T^\circ = \theta_N^\circ + x^3\theta_M^\circ$ is the initial temperature of the shell.

Since the heat conduction equations (2.1), (2.4), and (2.5) are of the same form for θ_N and θ_M, we need only discuss solutions for θ_N ; solutions for θ_M then follow merely by replacing the subscript N by M. We shall use the linear vector space of all functions θ_N such that

$$\int_0^\infty dt \int_R \theta_N^2 dS < \infty , \tag{2.6}$$

and the scalar product of two functions θ_N and θ_N' in this space is defined as

$$\int_0^\infty dt \int_R \theta_N \theta_N' \, dS, \tag{2.7}$$

where dS denotes the element of surface area on the shell middle surface.

Equation (2.1) contains the operator

$$L_N = \nabla^2 - \eta_N - \frac{1}{\kappa}\frac{\partial}{\partial t} \tag{2.8}$$

whose domain will be the set of all functions θ_N in the vector space such that $\partial\theta_N/\partial x_\alpha \partial x_\beta$ and $\partial\theta_N/\partial t$ are piecewise continuous and of integrable square and such that θ_N satisfies the zero initial condition $\lim_{t \to o} \theta_N = 0$ and the homogeneous boundary condition $\theta_N = 0$ on C. With the aid of Green's theorem for surfaces, we may establish the identity

$$\begin{aligned} \int_0^\infty dt \int_R \; (\theta_N^* L_N \theta_N - \theta_N L_N^* \theta_N^*) \; dS = \\ = \int_0^\infty dt \int_C n^\alpha \; (\theta_N^* \nabla_\alpha \theta_N - \theta_N \nabla_\alpha \theta_N^*) \; ds \\ + \frac{1}{\kappa}\int_R \left[\lim_{t \to o} \theta_N \theta_N^* - \lim_{t \to \infty} \theta_N \theta_N^* \right] dS, \end{aligned} \tag{2.9}$$

where n^α is the normal to C and

$$L_N^* = \nabla^2 - \eta_N + \frac{1}{\kappa}\frac{\partial}{\partial t} \tag{2.10}$$

is the operator adjoint to L_N whose domain is the same as that of L_N, except that instead of a zero initial condition, $\lim_{t' \to \infty} \theta_N^* = 0$. Thus, the right-hand side of (2.10) vanishes when θ_N and θ_N^* are in the domains of L_N and L_N^*, respectively. The definition of L_N may be extended to functions θ_N'' not in the domain of L_N by letting

$$\int_0^\infty dt \int_R \theta_N L_N \theta_N'' dS = \int_0^\infty dt \int_R \theta_N'' L_N^* \theta_N dS, \tag{2.11}$$

where $L_N \theta_N''$ may be a symbolic function.

We shall show that the Green's function for L_N is the actual or symbolic function $G_N(x,t;x',t')$ which satisfies

$$L_N G_N(x,t;x',t') = - \delta(x - x') \, \delta(t - t') \, , \tag{2.12a}$$

$$G_N(x,t;x',t') = 0 \qquad \text{if } t < t' \, , \tag{2.12b}$$

$$G_N(x,t;x',t') = 0 \, , \qquad x \text{ on } C \, , \tag{2.12c}$$

where $\delta(x - x')$ is the two-dimensional δ-function and $\delta(t - t')$ the one-dimensional δ-function.

Replacement of t by -t in (2.12) results in

$$L_N^* G_N^*(x,t;x'',t'') = -\delta(x-x'')\delta(t-t'') \tag{2.13a}$$

$$G_N^*(x,t;x'',t'') = 0 \quad \text{if } t \geq t'', \tag{2.13b}$$

$$G_N^*(x,t;x'',t'') = 0, \quad x \text{ on } C, \tag{2.13c}$$

where

$$G_N^*(x,t;x'',t'') = G_N(x,-t;x'',t'') \tag{2.14}$$

can be shown to be the Green's function for the adjoint operator L_N^*. Identification of θ_N with $G_N(x,t;x',t')$ and θ_N^* with $G_N^*(x,t;x'',t'')$ in (2.9) leads to the relation

$$G_N(x'',t'';x',t') = G_N^*(x',t';x'',t''), \tag{2.15}$$

while identification of θ_N' with $G_N^*(x,t;x',t')$ and use of (2.15) results in the following solution formula for θ_N in terms of its initial values, boundary values, and Q_N:

$$\begin{aligned}\theta_N(x,t) = &\int_0^t dt' \int_{R'} G_N(x,t;x',t')\, Q_N(x',t')dS' \\ &- \int_0^t dt' \int_{C'} n^\alpha(x')\nabla_\alpha G_N(x,t;x',t')\theta_N(x',t')ds' \\ &+ \int_{R'} G_N(x,t;x',0)\, \theta_N(x',0)dS'.\end{aligned} \tag{2.16}$$

A formula for θ_M follows by replacing subscript N by M in (2.16).

2.2 <u>A Solution Formula for Shallow Shells</u>. We recall that the stress differential equations of equilibrium for shallow shells may be written as

$$\nabla_\beta N^{\alpha\beta} + p^\alpha = 0, \tag{2.17a}$$

$$\nabla_\alpha \nabla_B\, z\, N^{\alpha\beta} + \nabla_\alpha Q^\alpha + p = 0\ , \tag{2.17b}$$

$$\nabla_\beta M^{\alpha\beta} - Q^\alpha = 0, \tag{2.17c}$$

where $N^{\alpha\beta}$, Q^α, $M^{\alpha\beta}$, p^α and p are the stress resultants, shear stress resultants, stress couple resultants, tangential surface tractions, and normal surface

traction, respectively; $\mathfrak{z}$ is the distance of the shell middle surface from a reference plane, and the covariant differentiation ∇_α now refers to a curvilinear coordinate system x^α in the reference plane. The strains $\gamma_{\alpha\beta}$ and changes in curvature $\kappa_{\alpha\beta}$ of the middle surface of the shallow shell are given by

$$2\gamma_{\alpha\beta} = \nabla_\beta v_\alpha + \nabla_\alpha v_\beta - 2\nabla_\alpha\nabla_\beta \mathfrak{z} w, \tag{2.18a}$$

$$\kappa_{\alpha\beta} = -\nabla_\alpha\nabla_\beta w, \tag{2.18b}$$

where v_α and w are the tangential and normal displacements of the middle surface of the shell.* The constitutive equations for isotropic homogeneous shallow shells, with the effect of thermal expansion included and the effect of transverse shear deformation neglected, take the form

$$N^{\alpha\beta} = A^{\alpha\beta\lambda\eta}\,\gamma_{\lambda\eta} - \frac{Eh}{(1-\nu)}\,a^{\alpha\beta}\,\alpha\theta_N, \tag{2.19a}$$

$$M^{\alpha\beta} = \frac{h^2}{12}A^{\alpha\beta\lambda\eta}\,\kappa_{\lambda\eta} - \frac{Eh^3}{12(1-\nu)}\,a^{\alpha\beta}\alpha\theta_M, \tag{2.19b}$$

where

$$A^{\alpha\beta\lambda\eta} = \frac{Eh}{(1-\nu^2)}\left[\nu a^{\alpha\beta}a^{\lambda\eta} + \frac{(1-\nu)}{2}\left(a^{\alpha\lambda}a^{\beta\eta} + a^{\alpha\eta}a^{\beta\lambda}\right)\right] \tag{2.20}$$

and Young's modulus E, Poisson's ratio ν and the coefficient of linear thermal expansion α are assumed independent of temperature as well as coordinates. Upon substitution of (2.18) and (2.19) into (2.17) we obtain the relation

$$Q^\alpha = \frac{h^2}{12}\,A^{\alpha\beta\lambda\eta}\nabla_\beta\kappa_{\lambda\eta} - \frac{Eh^3}{12(1-\nu)}\,\alpha a^{\alpha\beta}\nabla_\beta\theta_M, \tag{2.21}$$

and the displacement equations of equilibrium

$$A^{\alpha\beta\lambda\eta}\left[\nabla_\eta\nabla_\beta v_\lambda - \nabla_\beta(w\nabla_\lambda\nabla_\eta \mathfrak{z})\right] - \alpha' a^{\alpha\beta}\nabla_\beta\theta_N + p^\alpha = 0, \tag{2.22a}$$

$$- D\nabla^2\nabla^2 w + \nabla_\alpha\nabla_\beta \mathfrak{z} A^{\alpha\beta\lambda\eta}\left[\nabla_\eta v_\lambda - \nabla_\lambda\nabla_\eta \mathfrak{z}\, w\right] - \tag{2.22b}$$

---

* Marguerre (1938) and Green and Zerna (1954) employ displacements normal and parallel to the reference plane (axial and longitudinal displacements) which are related to the normal and tangential displacements w and v_α used here and in other works by w and $v_\alpha - \nabla_\alpha \mathfrak{z} w$, respectively, to within the approximations inherent in the theory of shallow shells. Also, the second fundamental form of the shell middle surface has the coefficient $b_{\alpha\beta} = \nabla_\alpha\nabla_\beta \mathfrak{z}$ to within the accuracy of shallow shell theory.

$$- \alpha' \, (\nabla^2 z\theta_N + \frac{h^2}{12}\nabla^2\theta_M) + p = 0, \tag{2.22b}$$

where

$$D = \frac{Eh^3}{12(1-\nu^2)} , \qquad \alpha' = \frac{Eh\alpha}{(1-\nu)} .$$

With the aid of (2.17) and (2.18) and the divergence theorem, it is easy to establish the relation

$$\int_R (N^{\alpha\beta}\gamma_{\alpha\beta} + M^{\alpha\beta}\kappa_{\alpha\beta})dS = \int_R (p^\alpha v_\alpha + pw)dS + I, \tag{2.23}$$

where

$$I = \int_C (N^{\alpha\beta}v_\beta - M^{\alpha\beta}\nabla_\beta w + Q^\alpha w)n_\alpha ds, \tag{2.24}$$

and n_α is now the normal to the curve C bounding the region R occupied by the projection of the shell middle surface onto the plane of the coordinates x^α. In the classical theory of shallow shells, since the effect of transverse shear deformation is neglected, the line integral in (2.24) should be written in an alternate form to reflect the fact that only four boundary conditions may be prescribed on C. Thus, following Green and Zerna (1954), we write the components of the couple resultant on C for directions normal and tangential to C as

$$H_N = M^{\alpha\beta}n_\alpha n_\beta, \qquad H_T = \epsilon_{\beta\lambda}M^{\alpha\beta}n_\alpha n^\lambda , \tag{2.25}$$

respectively, and for future convenience we do the same for the components of stress resultant and tangential displacement, i.e.,

$$\begin{aligned} N_N &= N^{\alpha\beta}n_\alpha n_\beta , & N_T &= \epsilon_{\beta\lambda}N^{\alpha\beta}n_\alpha n^\lambda , \\ v_N &= v_\beta n^\beta, & v_T &= \epsilon^{\beta\lambda}n_\lambda v_\beta , \end{aligned} \tag{2.26}$$

where $\epsilon_{\beta\lambda}$ and $\epsilon^{\beta\nu}$ are tensors of the ϵ-systems. Denoting the derivatives along the normal to C by $\partial/\partial n$ and $\partial/\partial s$, respectively, and defining the perhaps symbolic derivative** $\frac{\partial H_T}{\partial s}$ through

$$\int_C \frac{\partial H_T}{\partial s} w ds = -\int_C H_T \frac{\partial w}{\partial s} \; ds, \tag{2.27}$$

** See Friedman (1956), pp. 140-142.

(2.24) becomes

$$I = \int_C \left[N_N v_N - N_T v_T - H_N \frac{\partial w}{\partial n} + Vw \right] ds, \tag{2.28}$$

where

$$V = Q^{\alpha} n_{\alpha} + \frac{\partial H_T}{\partial s} \tag{2.29}$$

may be interpreted as the total shear resultant on C. If H_T has a jump discontinuity of magnitude ΔH_T at $s = s'$, then by (2.27) and (2.29)

$$V = Q^{\alpha} n_{\alpha} + \frac{\partial_o H_T}{\partial s} + \Delta H_T \delta(s-s'), \tag{2.30}$$

where $\frac{\partial_o}{\partial s}$ denotes differentiation in the ordinary (not symbolic) sense. The last term in (2.30) represents a concentrated load which is a well-known result in classical shell (and plate) theory. Appropriate boundary conditions for shallow shells are apparent from (2.28) and may be listed as

$$\begin{aligned}
&\text{either} \quad N_N = N_N^* &&\text{or} \quad v_N = v_N^*, \\
&\text{either} \quad N_T = N_T^* &&\text{or} \quad v_T = v_T^*, \\
&\text{either} \quad H_N = H_N^* &&\text{or} \quad \frac{\partial w}{\partial n} = \left(\frac{\partial w}{\partial n}\right)^*, \\
&\text{either} \quad V = V^* &&\text{or} \quad w = w^*,
\end{aligned} \tag{2.31}$$

on C, where the starred quantities are boundary values. In a later section we shall consider an example in which

$$N_N = N_N^*, \quad H_N = H_N^*, \quad v_T = w = 0, \quad \text{on C}, \tag{2.32}$$

and these boundary conditions will be used to illustrate the method of the present section. By (2.19), (2.23), and (2.28), the reciprocity theorem (Green's formula) for shallow shells is

$$\begin{aligned}
&\int_R \left[p'^{\alpha} v_{\alpha} + p'w + \alpha'(\theta'_N \gamma^{\lambda}_{\lambda} + \frac{h^2}{12}\theta'_M \kappa^{\lambda}_{\lambda}) \right] dS \\
&+ \int_C \left[N'_N v_N - N'_T v_T - H'_N \frac{\partial w}{\partial n} + V'w \right] ds \\
&= \int_R \left[p^{\alpha} v'_{\alpha} + pw' + \alpha'(\theta_N \gamma'^{\lambda}_{\lambda} + \frac{h^2}{12}\theta_M \kappa'^{\lambda}_{\lambda}) \right] dS \\
&+ \int_C \left[N_N v'_N - N_T v'_T - H_N \frac{\partial w'}{\partial n} + Vw' \right] ds ,
\end{aligned} \tag{2.33}$$

where primed and unprimed variables denote any two solutions of the shallow shell equations. The thermoelastic reciprocity theorem (2.33) may also be reached from the isothermal reciprocity theorem [$\theta_M = \theta_N = 0$ in (2.33)] by means of an analogy between the isothermal equations and the thermoelastic equations of shallow shells.$^+$ By (2.19) to (2.22), the analogy may be written

$$
\begin{aligned}
p_\circ^\alpha &= p^\alpha - \alpha' a^{\alpha\beta} \nabla_\alpha \theta_N , \\
p_\circ &= p - \alpha'(\nabla^2 z\theta_N + \frac{h^2}{12} \nabla^2 \theta_M) \\
N_\circ^{\alpha\beta} &= N^{\alpha\beta} + \alpha' a^{\alpha\beta} \theta_M , \\
M_\circ^{\alpha\beta} &= M^{\alpha\beta} + \frac{h^2}{12} \alpha' a^{\alpha\beta} \theta_M , \\
Q_\circ^\alpha &= Q^\alpha + \frac{h^2}{12} \alpha' a^{\alpha\beta} \nabla_\beta \theta_M ,
\end{aligned}
\tag{2.34}
$$

where subscript zero denotes a variable in the isothermal equations. As an aid in discussing solutions of (2.22) it is convenient to denote components of the middle surface displacement vector by U_i $(i = 1, 2, 3)$ where

$$U_\alpha = v_\alpha , \qquad (\alpha = 1,2), \qquad U_3 = w, \tag{2.35}^{++}$$

and consider the linear vector space of all U_i such that

$$\int_R (a^{\alpha\beta} v_\alpha v_\beta + w^2) dS < \infty . \tag{2.36}$$

The scalar product of two displacement vectors U_i and U_i' in this vector space is defined as

$$\int_R (a^{\alpha\beta} v_\alpha v_\beta' + ww') dS . \tag{2.37}$$

$^+$The analogy is similar to that given by Duhamel (1838) for three-dimensional thermoelasticity, and the Betti reciprocity theorem has been extended to thermoelasticity with the aid of this analogy by Goodier (1958).

$^{++}$The only tensor properties of U_i are those of its components v_α and w which are a vector and scalar, respectively, with reference to transformations of coordinates in the reference plane.

In (2.22) there appears the operator $\mathcal{L}^{ij}$ defined by

$$\mathcal{L}^{\alpha\lambda} = A^{\alpha\beta\lambda\eta} \nabla_\beta \nabla_\eta ,$$

$$\mathcal{L}^{\alpha 3} = -A^{\alpha\beta\lambda\eta} (\nabla_\beta \nabla_\lambda \nabla_\eta z + \nabla_\lambda \nabla_\eta z \nabla_\beta),$$

$$\mathcal{L}^{3\lambda} = A^{\alpha\beta\lambda\eta} \nabla_\alpha \nabla_\beta z \nabla_\eta , \tag{2.38}$$

$$\mathcal{L}^{33} = -D\nabla^2\nabla^2 - A^{\alpha\beta\lambda\eta} \nabla_\alpha \nabla_\beta z \nabla_\lambda \nabla z$$

whose domain will be the set of all U_i in the vector space such that second partial derivatives of U_α and fourth partial derivatives of U_3 are piecewise continuous and of integrable square and such that U_i meets homogeneous boundary conditions on C of the form (2.32) with $N^*_N = H^*_N = 0$. Then (2.22) may be written as

$$\mathcal{L}^{ij} U_j - \Theta^i + P^i = 0, \tag{2.39}$$

where

$$P^\alpha = p^\alpha, \qquad P^3 = p, \qquad \Theta^\alpha = \alpha' a^{\alpha\beta} \nabla_\beta \theta_N ,$$

$$\Theta^3 = \alpha'(\nabla^2 z \theta_N + \frac{h^2}{12} \nabla^2 \theta_M). \tag{2.40}$$

By (2.39), the reciprocity theorem (2.33) becomes

$$\int_R \left[(-\mathcal{L}^{ij} U'_j + \Theta'^i) U_i + \mathfrak{n}^i(\theta') U_i \right] dS + I(U', U)$$

$$= \int_R \left[(-\mathcal{L}^{ij} U_j + \Theta^i) U'_i + \mathfrak{n}^i(\theta) U_i \right] dS + I(U, U'), \tag{2.41}$$

where $\mathfrak{n}^i(\theta)$ is the operator defined by

$$\mathfrak{n}^\alpha(\theta) = \alpha' \theta_N a^{\alpha\beta} \nabla_\beta ,$$

$$\mathfrak{n}^3(\theta) = -\alpha'(\theta_N \nabla^2 z + \frac{h^2}{12} \theta_M \nabla^2) , \tag{2.42}$$

and

$$I(U', U) \quad \int_C [N'_N v_N - N'_T v_T - H'_N \frac{\partial w}{\partial n} + V' w] ds . \tag{2.43}$$

Setting $\theta_N = \theta_M = 0$ in (2.41), $\mathcal{L}^{ij}$ is seen to be self-adjoint and may be

extended to functions U_i'' not in its domain by the definition

$$\int_R U_i \mathcal{L}^{ij} U_j'' \, dS = \int_R U_i'' \mathcal{L}^{ij} U_j \, dS. \tag{2.44}$$

We shall show that the Green's function for $\mathcal{L}^{ij}$ is the perhaps symbolic (tensorial) function $G_{ij}(x;x')$, which satisfies the equation

$$\mathcal{L}^{ij} G_{jk}(x;x') = -\delta_k^i \, \delta(x-x') \ , \tag{2.45}$$

and homogeneous boundary conditions* corresponding to (2.32) which with the aid of (2.19) and (2.26) may be written as

$$\begin{aligned} N_{Ni} &= n_\alpha n_\beta \, A^{\alpha\beta\lambda\eta} \, [\nabla_\lambda G_{\alpha i}(x;x') - G_{3i}(x;x') \nabla_\lambda \nabla_\eta z] = 0 \ , \\ H_{Ni} &= -n_\alpha n_\beta \frac{h^2}{12} A^{\alpha\beta\lambda\eta} \, \nabla_\lambda \nabla_\eta \, G_{3i}(x;x') = 0 \ , \\ v_{Ti} &= n^\alpha \, G_{\alpha i}(x;x') = 0 \ , \qquad G_{3i}(x;x') = 0 \ , \end{aligned} \tag{2.46}$$

for x on C. In (2.41), identification of U_j with G_{jk} $(x;x')$ and U_j' with $G_{jl}(x;x'')$ followed by application of (2.45) and (2.46) yields the relation

$$G_{kl}(x';x'') = G_{lk}(x'',x') \ . \tag{2.47}$$

Similarly, identification of U_j' with $G_{jk}(x;x')$ in (2.41) and application of (2.32), (2.39), (2.45), (2.46), and (2.47) results in

$$\begin{aligned} U_{i(x)} &= \int_{R'} [G_{ij}(x;x') \, P^j(x') + \Pi'^j \, (\theta(x')) \, G_{ij}(x;x')] \, dS' \\ &+ \int_C [n^\beta(x') G_{i\beta}(x;x') N_N^*(x') - \frac{\partial}{\partial n'} \, G_{i3}(x;x') \, H_N^*(x')] \, ds'. \end{aligned} \tag{2.48)**}$$

---

*It should be noted that under some boundary conditions in the original problem, the boundary conditions on the Green's function will not be homogeneous. For example, if N_N and N_T are prescribed on C in the original problem, corresponding homogeneous boundary conditions on the Green's function would not be possible in view of over-all equilibrium requirements. In such cases the Green's function must satisfy inhomogeneous boundary conditions which may conveniently be taken in a form similar to rigid body displacements as done for a flat plate by Bergman and Schiffer (1953, p. 239).

**The primed operators act on the x'^α variables of the Green's function.

The solution formula (2.48) expresses the displacements in terms of the surface and edge tractions and the temperature resultant field. In the absence of surface and edge tractions (2.48) may be written as

$$U_i(x) = \int_{R'} [\tilde{U}_i^N (x;x')\theta_N(x') + \tilde{U}_i^M (x;x')\theta_M(x')]\, dS', \tag{2.49}$$

where by (2.42)

$$\begin{aligned} \tilde{U}_i^N (x;x') &= \alpha' \, [a^{\lambda\beta} (x')\nabla'_\beta \, G_{i\lambda}(x;x') - \nabla'^2 z \,(x')\, G_{i3}(x;x')] \,, \\ \tilde{U}_i^M (x;x') &= -\frac{h^2}{12}\alpha \nabla'^2 \, G_{i3}(x;x') \,. \end{aligned} \tag{2.50}$$

By manipulation of (2.45) it is easy to show that $\tilde{U}_i^N (x;x')$ satisfies (2.22) with $\theta_N = \delta(x-x')$ and $\theta_M = p^\alpha = p = 0$, i.e.,

$$\begin{aligned} \mathcal{L}^{\alpha j} \tilde{U}_j^N (x;x') &= \alpha' a^{\alpha\beta} \nabla_\beta \, \delta(x-x'), \\ \mathcal{L}^{3j} \tilde{U}_j^N (x;x') &= \alpha' \nabla^2 z \, \delta(x-x'), \end{aligned} \tag{2.51a}$$

while $\tilde{U}_i^M (x;x')$ satisfies (2.22) with $\theta_M = \delta(x-x')$ and $\theta_N = p^\alpha = 0$, i.e.,

$$\begin{aligned} \mathcal{L}^{\alpha j}\tilde{U}_j^N (x;x') &= 0, \\ \mathcal{L}^{3j}\tilde{U}_j^M (x;x') &= \frac{h^2}{12}\alpha' \nabla^2 \delta(x-x'). \end{aligned} \tag{2.51b}$$

Further, by (2.46), $\tilde{U}_i^N$ satisfies the homogeneous boundary conditions

$$\begin{aligned} &n_\alpha n_\beta A^{\alpha\beta\lambda\eta} \, (\nabla_\lambda \tilde{U}^N(x;x') - \tilde{U}_3^N \, (x;x') \, \nabla_\lambda \nabla_\eta z \,) = 0, \\ &n_\alpha n_\beta A^{\alpha\beta\lambda\eta} \, \nabla_\lambda \nabla_\eta \tilde{U}_3^N = 0, \\ &n^\alpha \tilde{U}_\alpha^N \,(x;x') = \tilde{U}_3^N \,(x;x') = 0, \qquad x \text{ on } C, \end{aligned} \tag{2.52}$$

and identical boundary conditions are met by $\tilde{U}_i^M$. The stress and couple resultants may now be obtained from (2.19) and (2.48) or(2.49).

2.3 <u>A Solution Formula for Combined Thermoelastic Problems of Shallow Shells.</u> The solution formula (2.16) for heat conduction and the elastostatic solution formula (2.49) may be combined and after interchange of the order of integration written as

$$
\begin{aligned}
U_i(x,t) = \int_0^t dt' \int_{R'} \Big[& \mathcal{Y}_i^N (x,t;x';t')\, Q_N(x',t') \\
& + \mathcal{Y}_i^M (x,t;x',t')\, Q_M(x',t') \Big] dS' - \int_0^t dt' \\
\int_C n^\alpha(x') \Big[& \nabla'_\alpha \mathcal{Y}_i^N (x,t;x',t')\, \theta_N^*(x',t') + \nabla'_\alpha \mathcal{Y}_i^M (x,t;x',t') \theta_M^*(x',t') \Big] ds' \\
+ \int_{R'} \Big[& \mathcal{Y}_i^N (x,t;x',0)\, \theta_N^\circ(x') + \mathcal{Y}_i^M (x,t;x',0)\, \theta_M^\circ(x') \Big] dS',
\end{aligned}
\tag{2.53}
$$

where

$$
\mathcal{Y}_i^N (x,t;x'',t'') = \int_{R'} \tilde{U}_\alpha^N (x;x')\, G_N(x',t;x'',t'')\, dS', \tag{2.54}
$$

and similarly for $\mathcal{Y}_i^M$. The truth of (2.53) is easily verified, since by (2.51a) and (2.54)

$$
\begin{aligned}
\mathcal{L}^{\alpha j} \mathcal{Y}_j^N (x,t;x'',t'') &= \alpha' \, a^{\alpha\beta} \nabla_\beta \dot{G}_N (x,t;x'',t''), \\
\mathcal{L}^{3j} \mathcal{Y}_j^N (x,t;x'',t'') &= \alpha' \, \nabla^2 z \, G_N (x,t;x'',t''), \\
\mathcal{L}^{\alpha j} \mathcal{Y}_j^M (x,t;x'',t'') &= 0 \quad , \\
\mathcal{L}^{3j} \mathcal{Y}_j^M (x,t;x'',t'') &= \frac{h^2}{12} \alpha' \, \nabla^2 G_M (x,t;x'',t''),
\end{aligned}
\tag{2.55}
$$

and then by (2.53) and (2.16)

$$
\begin{aligned}
\mathcal{L}^{\alpha j} U_j (x,t) &= \alpha' a^{\alpha\beta} \nabla_\beta \, \theta_N (x,t), \\
\mathcal{L}^{3j} U_j (x,t) &= \alpha' \left[\nabla^2 z \, \theta_N(x;t) + \frac{h^2}{12} \nabla^2 \theta_M(x,t) \right]
\end{aligned}
$$

which are precisely the field equations (2.39) with $P^i = 0$. By (2.54), $\mathcal{Y}_i^N$ and $\mathcal{Y}_i^M$ satisfy boundary conditions of the form (2.52). Thus, $\mathcal{Y}_j^N$ and $\mathcal{Y}_j^M$ play the role of Green's function for combined quasi-static thermoelastic problems in the theory of shallow shells and (2.53) determines the middle surface displacement in terms of the surface or internal heating, edge temperature, and initial temperature. By (2.16), (2.19), and (2.53) the stress resultants are

$$N^{\alpha\beta}(x,t) = \int_0^t dt' \int_{R'} \Big[N_N^{\alpha\beta}(x,t;x',t')\, Q_N(x',t')$$

$$+ N_M^{\alpha\beta}(x,t;x',t')\, Q_M(x',t') \Big] dS'$$

$$- \int_0^t dt' \int_{C'} n^{\alpha}(x') \Big[\nabla_\alpha N_N^{\alpha\beta}(x,t;x',t')\, \theta_N(x',t')$$

$$+ \nabla'_\alpha N_M^{\alpha\beta}(x,t;x',t')\, \theta_M(x',t') \Big] ds' \qquad (2.57)$$

$$+ \int_{R'} \Big[N_N^{\alpha\beta}(x,t;x',0)\, \theta_N^{\circ}(x')$$

$$+ N_M^{\alpha\beta}(x,t;x',0)\, \theta_M^{\circ}(x') \Big] dS' ,$$

where

$$N_N^{\alpha\beta}(x,t;x',t') = A^{\alpha\beta\lambda\eta} \Big[\nabla_\lambda \mathcal{G}_\eta^N(x,t;x',t')$$

$$- \mathcal{G}_3^N(x,t;x',t') \nabla_\lambda \nabla_\eta\, z(x) \Big] - \alpha' a^{\alpha\beta}(x)\, G^N(x,t;x',t'), \qquad (2.58)$$

and similarly for $N_M^{\alpha\beta}$. By (2.58), $N_N^{\alpha\beta}$ is just the resultant obtained directly from the Green's functions by an equation of the form (2.19a). Similar expressions may be obtained for the couple resultants and shear stress resultants. The question of existence of solution U_i for specified edge temperature, initial temperature, and heating will not be considered here, although some restrictions on the region R and on the specified functions certainly will be necessary.

2.4 Remarks on Extension of the Results to General Theory of Shells. The method developed in the foregoing may also be applied in the more general theory of shells which are not necessarily shallow and in which the effect of transverse shear deformation may also be included. The results obtained for shallow shells apply immediately to the simplified theory (sometimes referred to as the "technical" theory) of shells, presented, e.g., by Green and Zerna (1954), the only change being that x^α are regarded as middle surface coordinates rather than coordinates in a reference plane. With only slight modification, the formulas of this section also extend to the version of shell theory commonly known as Love's first approximation, + in which the constitutive

+There are at least three different versions of Love's approximation in the literature. We refer here to the one given in vectorial form by Reissner (1941). A full discussion and additional references are given by Naghdi (1962).

equations may be written as

$$
\begin{aligned}
N^{\beta\alpha} &= A^{\alpha\beta\lambda\eta}\,(\nabla_\lambda v_\eta - b_{\lambda\eta} w) - \alpha' a^{\alpha\beta}\theta_N\ , \\
M^{\beta\alpha} &= -\frac{h^2}{12} A^{\alpha\beta\lambda\eta}\,(\nabla_\lambda \nabla_\eta w + \nabla_\lambda\ (b^\gamma_\eta v_\gamma)\ ) - \frac{h^2}{12}\alpha'\ a^{\alpha\beta}\theta_M\ ,
\end{aligned}
\tag{2.59}
$$

where the covariant differentiation now refers to a curvilinear coordinate system on the shell middle surface and $A^{\alpha\beta\lambda\eta}$ is given by (2.20) with $a^{\alpha\beta}$ now being the conjugate metric for the middle surface coordinate system. When (2.59) is combined with the general equilibrium equations for shells we have

$$
\begin{aligned}
& A^{\alpha\beta\lambda\eta}\,\nabla_\beta(\nabla_\eta v_\lambda - b_{\lambda\eta} w) + \frac{h^2}{12} b^\alpha_\rho\, A^{\rho\beta\lambda\eta}\,\nabla_\beta\Big[\nabla_\lambda\nabla_\eta\, w \\
& + \nabla_\lambda(b^\gamma_\eta\ v_\gamma)\Big] - \alpha'\left[a^{\alpha\beta}\,\nabla_\beta\theta_N - \frac{h^2}{12}\, b^{\alpha\beta}\,\nabla_\beta\theta_M\right] + p^\alpha = 0\ , \\
& -\frac{h^2}{12} A^{\alpha\beta\lambda\eta}\,\nabla_\alpha\nabla_\beta\left[\nabla_\lambda\nabla_\eta w + \nabla_\lambda\ (b^\gamma_\eta\ v_\gamma)\right] \\
& + A^{\alpha\beta\lambda\eta}\, b_{\alpha\beta}\,(\nabla_\eta v_\lambda - b_{\lambda\eta} w) - \alpha'\left[\frac{h^2}{12}\nabla^2\theta_M + b^\alpha_\alpha\ \theta_N\right] + p = 0
\end{aligned}
\tag{2.60}
$$

which are of the form (2.39) but with the operator $\mathcal{L}^y$ and the temperature term Θ^i no longer given by (2.38) and (2.40). Also, it is easily verified that the form (2.41) remains unchanged. Since the heat conduction equations (2.1) with solution formula (2.16) are valid to the same degree of approximation as (2.59), it follows that the result (2.53) holds also for Love's first approximation. However, the Green's function $\mathcal{G}^N_j$ (x,t;x',t') will now be a solution of (2.60) operated on by L_N with $L_N\theta_N = \infty\delta(x-x')\ \delta(t-t')$, $\theta_M = p = p^\alpha = 0$ and under appropriate boundary conditions. A similar statement holds for $\mathcal{G}^M_j$ and also for G_{ij} in (2.48). In connection with Love's first approximation, it may be recalled that integral formulas for displacements and their derivatives for the isothermal case are given by Naghdi (1960); these formulas are contained in (2.48) and its derivatives. It should be remembered, however, that certain inconsistencies are present in the constitutive equations (2.59) in that $N^{\beta\alpha}$ and $M^{\beta\alpha}$ do not vanish identically under rigid body displacement, nor do they satisfy the equation of moment equilibrium about a normal to the shell middle surface, except for spherical shells.[++] While the degree of error is of the order $\frac{h}{R}$ (R being

[++]The inconsistencies mentioned are also present in the constitutive equations of shallow shell theory, although they are not considered serious in view of the assumption of shallowness.

the minimum principal radius of curvature), and $\frac{h}{R} \ll 1$ is a basic postulate of thin shell theory, nevertheless difficulties have been encountered in some problems and at least on theoretical grounds improved constitutive equations are desirable. To obtain improved theories, additional higher order geometrical terms have been retained in numerous derivations of constitutive equations and the (often) equally important effect of transverse shear deformation has sometimes also been included. A more complete discussion and a detailed comparison of various constitutive equations is given by Naghdi (1962). When more exact constitutive equations are applied to thermoelastic problems of shells, the use of heat conduction equations of corresponding accuracy seems desirable. Thus, the heat conduction equations for thin shells derived by Bolotin (1960) could be improved by retaining geometrical quantities of higher order in h/R, in which case it may also be necessary to modify the assumption (2.3). Further, the temperature resultants θ_N and θ_M may not enter the constitutive equations in such a simple manner as in (2.59).* While additional effort may be required to extend the present method to thermoelastic problems in improved theory of shells, the main ideas of the Green's function approach will still apply provided a reciprocity (Green's) formula is available in simple form, as is the case for a system of improved constitutive equations given by Naghdi (1962). In fact, improved equations due to Flügge for circular cylindrical shells are employed by Goodier and Nevill (1961) to obtain formulas somewhat analagous to (2.49), and they also derive similar formulas for various specific problems in theories of thin bars, thin plates, membrane shells, and three-dimensional elasticity. In these as well as other problems, it should be possible to combine the integral formulas for displacements in terms of the temperature field with integral formulas for solution of appropriate heat conduction equations, thereby obtaining formulas for displacements in the combined thermoelastic problem as in the foregoing treatment for shallow shells.

3. APPLICATION OF THE QUASI-STATIC THEORY: THERMOELASTIC GREEN'S FUNCTIONS FOR TWO CLASSES OF SHALLOW SHELLS

3.1 A Class of Unlimited Shallow Shells. First we shall obtain the Green's functions G_N and G_M for temperature resultants in an unlimited shallow shell.

*Note, however, that the θ_N and θ_M terms in (2.59) do meet the equation of moment equilibrium about a normal to the middle surface.

These Green's functions are independent of the form of the middle surface, since Z does not enter (2.12). Although G_N and G_M could be obtained from the known Green's function for the two-dimensional heat conduction equation by a simple transformation, for illustrative purposes they are derived here following the method established by Friedman (1956, p.293) since this same method will be applied later to obtain the thermoelastic Green's functions $\mathcal{G}_i^N$ and $\mathcal{G}_i^M$. Thus, we may write (2.12a) as

$$G_N(x,t;x',t') = \frac{-1}{\left(\nabla^2 - \eta_N - \frac{1}{\kappa}\frac{\partial}{\partial t}\right)} \delta(x - x')\,\delta(t - t')\,, \qquad (3.1)$$

and interpret the result with the aid of appropriate spectral representations for the Laplacian (in the entire plane) and $\frac{\partial}{\partial t}$ (on $0 \leq t \leq \infty$ with zero initial condition), namely**

$$\delta(x - x') = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\xi_1(x_1-x_1')}\, e^{i\xi_2(x_2-x_2')}\, d\xi_1 d\xi_2\,, \qquad (3.2)$$

and

$$\delta(t - t_\circ) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} e^{s(t-t')}\, ds\,, \qquad (3.3)$$

respectively, where x_1 and x_2 are rectangular Cartesian coordinates. By (3.1) to (3.3) we have

$$G_N(x,t;x',t') = \frac{\kappa}{8\pi^3 i}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{a-i\infty}^{a+i\infty} \frac{e^{s(t-t')}\, e^{i\xi_1(x_1-x_1')}\, e^{i\xi_2(x_2-x_2')}}{s + \kappa(\eta_N + \xi_1^2 + \xi_2^2)}\, ds\, d\xi_1 d\xi d_2 \qquad (3.4)$$

which, upon introducing

$$\frac{1}{s + \kappa(\eta_N + \xi_1^2 + \xi_2^2)} = \int_0^{\infty} \exp\left[-ts - t\kappa(\eta_N + \xi_1^2 + \xi_2^2)\right] dt, \qquad (3.5)$$

$$r^2 = (x_1 - x_1')^2 + (x_2 - x_2')^2\,,$$

after some elementary manipulations, becomes

$$G_N(x,t;x',t') = \frac{1}{4\pi(t-t')} \exp\left[\frac{-r^2}{4\kappa(t-t')} - \kappa\eta_N\,(t-t')\right],\quad t - t' \geq 0,$$
$$= 0,\quad t - t' < 0, \qquad (3.6)$$

and a similar equation holds for $G_M(x,t;x',t')$. Next, we consider the class of unlimited shallow shells with middle surfaces of arbitrary quadratic form which by an orthogonal transformation of coordinates may be brought to

---

**As noted by Friedman (1956), (3.2) and (3.3) contain the two-dimensional Fourier transform theorem and the Laplace transform theorem, respectively.

$$\mathfrak{z} = \frac{1}{2}\left(b_1 x_1^2 + b_2 x_2^2\right) , \qquad (3.7)^{+}$$

where b_1 and b_2 are the principal curvatures of the middle surface of the shallow shell. For the class (3.7), Marguerre's (1938) formulation for elastostatic problems of shallow shells in terms of w and an Airy stress function F, in the absence of surface tractions and the presence of temperature resultants, reads

$$\begin{aligned} &\nabla^2\nabla^2 F + Eh\,(b_2L_1 + b_1L_2)\,w + Eh\,\alpha\nabla^2\theta_N = 0 , \\ &D\,\nabla^2\nabla^2 w - (b_2L_1 + b_1L_2)\,F + (1-\nu)\,D\,\alpha\nabla^2\theta_M = 0 , \end{aligned} \qquad (3.8a)$$

where

$$L_1 = \frac{\partial^2}{\partial x_1{}^2} , \qquad L_2 = \frac{\partial^2}{\partial x_2{}^2} , \qquad (3.8b)$$

and the stress resultants are

$$N_{11} = L_2F , \; N_{22} = L_1F, \quad N_{12} = -\frac{\partial^2 F}{\partial x_1 \partial x_2} . \qquad (3.9)$$

By (2.18), (2.19), (3.7), and (3.9) the tangential components of middle surface displacement to within a rigid body displacement may be written as

$$v_1 = \int_0^{x_1} \left[\frac{1}{Eh}\,(L_2F - \nu L_1F) + b_1 w + \alpha\theta_N\right] dx_1, \qquad (3.10)$$

and an expression for v_2 given by (3.10) with subscripts 1 and 2 interchanged. To obtain the thermoelastic Green's function $\mathcal{L}_i^N\,(x,t;x',t')$, we replace F, w, θ_N, and θ_M in (3.8) by $F^N, \mathcal{L}_3{}^N$, G_N, and 0, respectively, whence, with the aid of (3.1), (3.8) yields

$$\begin{aligned} \mathcal{L}_3^N &= \frac{k^2\alpha(b_2L_1 + b_1L_2)\,\nabla^2}{[\nabla^8 + k^2(b_2L_1 + b_1L_2)^2][\nabla^2 - \eta_N - \frac{1}{\kappa}\frac{\partial}{\partial t}]}\;\delta(x-x')\,\delta(t-t') , \\ F^N &= \frac{Eh\,\alpha\,\nabla^6}{\left[\nabla^8 + k^2(b_2L_1 + b_1L_2)^2\right]\left[\nabla^2 - \eta_N - \frac{1}{\kappa}\frac{\partial}{\partial t}\right]}\;\delta(x-x')\,\delta(t-t') , \end{aligned} \qquad (3.11)$$

where

$$k^2 = \frac{Eh}{D} = \frac{12}{h^2}\,(1 - \nu^2) .$$

+For a shallow shell with middle surface analytic at the origin, (3.7) represents the first significant term of the series expansion of $\mathfrak{z}$ in the neighborhood of the origin.

This result may be interpreted using (3.3) and the spectral representation

$$\delta(x_1 - x_1') = \frac{1}{2\pi}\int_{\infty}^{+\infty} \ell^{i\xi_1(x_1 - x_1')}\, d\xi_1 \tag{3.12}$$

for the operator L and a similar representation for L_2. Thus, following the procedure used to obtain (3.6), we find that

$$\begin{aligned} \mathcal{G}_3^N &= \frac{k\kappa d}{8\pi^2 i}\, \ell^{-\kappa\eta_N(t-t')}\, (I_1 - I_2)\ , \\ F^N &= \frac{Eh\kappa d}{8\pi^2}\, \ell^{-\kappa\eta_N(t-t')}\, (I_1 + I_2)\ , \end{aligned} \tag{3.13}$$

where

$$\left\{\begin{matrix} I_1 \\ I_2 \end{matrix}\right\} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{(\xi_1^2 + \xi_2^2)\, \ell^{-\kappa(\xi_1^2 + \xi_2^2)(t-t_o)}}{(\xi_1^2 + \xi_2^2) \pm ik(b_2\xi_1{}^2 + b_1\xi_2^2)} \tag{3.14}$$

$$\ell^{i\xi_1(x_1 - x_1')}\, \ell^{i\xi_2(x_2 - x_2')}\, d\xi_1\, d\xi_2$$

are complex conjugate functions. The component $\mathcal{G}_1^N$ of the Green's function by (3.10), (3.4), and (3.13) is

$$\mathcal{G}_1^N = \frac{-\kappa\alpha}{4\pi^2 i}\, \ell^{-\kappa\eta_N(t-t')} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{\xi_1}\left[\ell^{i\xi_1(x_1-x_1')} - \ell^{-i\xi_1 x_1'}\right]$$

$$\left[\frac{(\xi_1^2 + \xi_2^2)^3(\xi_2^2 - \nu\xi_1^2) + b_1k^2\,(\xi_1{}^2 + \xi_2^2)(b_2\xi_1^2 + b_1\xi_2^2)}{(\xi_1{}^2 + \xi_2^2)^4 + k^2\,(b_2\xi_1^2 + b_1\xi_2{}^2)^2} - 1\right] \tag{3.15}$$

$$\ell^{-\kappa(\xi_1^2 + \xi_2^2)(t-t')}\, \ell^{i\xi_2(x_2 - x_2')}\, d\xi_1\, d\xi_2\ ,$$

and $\mathcal{G}_2^N$ $(x,t;x',t')$ is given by (3.15) with subscripts 1 and 2 interchanged. Following the same procedure as before, the Green's functions $\mathcal{G}_i^M$ are found to be

$$\mathcal{G}_3^M = \frac{(1+\nu)\kappa\,\alpha}{8\pi^2}\, \ell^{-\kappa\eta_M(t-t')}\, (I_1 + I_2)\ ,$$

$$\mathcal{G}_1^M = -\frac{(1+\nu)\,\kappa\alpha}{4\pi^2 i}\, \ell^{-\kappa\eta_M(t-t')} \int_{\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{\xi_1}\left[\ell^{i\xi_1(x_1-x_1')} - \ell^{-i\xi_1 x'}\right]$$

$$\left[\frac{(\xi_1^2 + \xi_2^2)(b_2\xi_1^2 + b_1\xi_2^2)(\xi_2^2 - \nu\xi_1^2) - b_1(\xi_1^2 + \xi_2^2)^3}{(\xi_1^2 + \xi_2^2)^4 + k^2(b_2\xi_1^2 + b_1\xi_2{}^2)^2} - 1\right] \tag{3.16}$$

$$\ell^{-\kappa(\xi_1^2 + \xi_2^2)(t-t')}\, \ell^{i\xi_2(x_2-x_2')}\, d\xi_1\, d\xi_2\ ,$$

a similar expression for $\mathcal{G}_2^M(x,t;x',t')$, and

$$F^M = -\frac{(1+\nu)Eh\ \kappa\alpha}{8\pi^2 ki}\ \ell^{-\kappa\eta_M(t-t')}\ (I_1 - I_2). \tag{3.17}$$

Next, in order to study the integrals I_1 and I_2 defined by (3.14) we introduce polar coordinates through

$$\begin{aligned} \xi_1 &= \rho\cos\phi\ , & \xi_2 &= \rho\sin\phi\ , \\ x_1 - x_1' &= r\cos\theta\ , & x_2 - x_2' &= r\sin\theta\ , \end{aligned} \tag{3.18}$$

and then

$$\left\{\begin{matrix} I_1 \\ I_2 \end{matrix}\right\} \int_0^\infty \int_{\phi_0}^{\phi_0 + 2\pi} \frac{\ell^{-\kappa\rho^2(t-t')}\ \ell^{ir\rho\cos(\phi-\theta)}\ \rho d\phi d\rho}{\rho^2 \pm k\ (b_1\sin^2\phi + b_2\cos^2\phi)}\ . \tag{3.19}$$

Expansion of the denominator of (3.19) in Fourier cosine series yields

$$\left\{\begin{matrix} I_1 \\ I_2 \end{matrix}\right\} \int_0^\infty \int_{\phi_0}^{\phi_0 + 2\pi} \sum_{m=0}^{\infty} a_m\ (\pm k,\rho^2)\ \ell^{-\kappa\rho^2(t-t')}$$

$$\ell^{ir\rho\cos(\phi-\theta)}\ \cos 2m\phi\ \rho d\phi d\rho\ , \tag{3.20a}$$

where

$$\begin{aligned} a_0\ (\pm k,\rho^2) &= \frac{1}{\pi}\int_0^\pi \frac{d\phi}{\rho^2 \pm ik\ (b_1\sin^2\phi + b_2\cos^2\phi)}, \\ a_m\ (+ k,\rho^2) &= \frac{2}{\pi}\int_0^\pi \frac{\cos 2m\phi\, d\phi}{\rho^2 \pm ik\ (b_1\sin^2\phi + b_2\cos^2\phi)}\ . \end{aligned} \tag{3.20b}$$

Evaluation of the definite integrals in (3.20b) is carried out in the Appendix where it is found that

$$\begin{aligned} a_0\ (\pm k,\rho^2) &= \frac{1}{\lambda_1\lambda_2}\ , \\ a_m\ (\pm k,\rho^2) &= \frac{\beta^m + \beta^{-m}}{\lambda_1\lambda_2} - \frac{4}{(\lambda_1^2 - \lambda_2^2)} \sum_{k=1}^{m-1} \beta^{m-2k-1}\ , \\ &\qquad \text{for } m \geqq 1,\ \lambda_1 \neq \lambda_2\ , \\ a_m\ (\pm k,\rho^2) &= 0\ , \qquad \text{for } m \geqq 1,\ \lambda_1 = \lambda_2\ , \end{aligned} \tag{3.21a}$$

with the notation

$$\lambda_1 = (\rho^2 \pm ikb_1)^{\frac{1}{2}} , \qquad \lambda_2 = (\rho^2 \pm ikb_2)^{\frac{1}{2}} ,$$
$$\beta = (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)^{-1} . \tag{3.21b}$$

Recalling [Watson (1944)] that the Bessel function of first kind may be represented by the integral

$$J_n(n\rho) = \frac{i^{-n}}{2\pi} \int_0^{2\pi} \ell^{ir\rho\cos\phi} \cos n\phi d\phi ,$$

and taking $\phi_0 = \theta$, (3.20a) reduces to

$$\begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} = 2\pi \sum_{m=0}^{\infty} (-1)^m \cos 2m\theta \int_0^{\infty} a_m (\pm k, \rho^2) \ell^{-\kappa\rho^2(t-t')} J_{2m}(r\rho) \rho d\rho \tag{3.22}$$

which appears suited to numerical evaluation. At $r=0$, by (3.20) and with $\rho' = \rho^2$, (3.22) becomes

$$\begin{Bmatrix} I_1 \\ I_2 \end{Bmatrix} = \pi \int_0^{\infty} \frac{\ell^{-\rho'\kappa(t-t')} d\rho'}{(\rho' \pm ikb_1)^{\frac{1}{2}} (\rho' \pm ikb_2)^{\frac{1}{2}}} \tag{3.23}$$

which is in the form of a Laplace transform and will be evaluated for several specific shell geometries of the class (3.7). For an unlimited spherical shell[++] (of radius R)

$$b_1 = b_2 = -\frac{1}{R} , \tag{3.24}$$

and by (3.23) and a known Laplace transform [Erdélyi (1954)] we have at $r = 0$

$$I_1 + I_2 = 2\pi [-Ci(\tau) \cos\tau - Si(\tau) \sin\tau + \tfrac{\pi}{2} \sin\tau] ,$$
$$I_1 - I_2 = 2\pi i [Ci(\tau) \sin\tau - Si(\tau) \cos\tau + \tfrac{\pi}{2} \cos\tau] , \tag{3.25}$$
$$\tau = \frac{k\kappa}{R}(t-t') ,$$

where $Si(\tau)$ and $Ci(\tau)$ are the sine and cosine integrals defined as usual by

[++]The unlimited shallow spherical shell is actually an unlimited parabaloid of revolution which in the neighborhood of the apex may be regarded as equivalent to a spherical shell with edge sufficiently far removed from the shallow region of interest.

$$\mathrm{Si}(\tau) = \int_0^{\tau} \frac{\sin y}{y}\, dy\,, \qquad \mathrm{Ci}(\tau) = -\int_{\tau}^{\infty} \frac{\cos y}{y}\, dy. \tag{3.26}$$

Also, for the shallow spherical shell, (3.21) and (3.22) result in

$$\begin{aligned} I_1 + I_2 &= 2\pi \int_0^{\infty} \frac{\ell^{-\kappa\rho'(t-t')} J_0(\kappa\sqrt{\rho'})\, \rho'\, d\rho'}{\rho'^2 + \frac{k^2}{R^2}}\,, \\ I_1 - I_2 &= \frac{2\pi i k}{R} \int_0^{\infty} \frac{\ell^{-\kappa\rho'(t-t')} J_0(\kappa\sqrt{\rho'})\, d\rho'}{\rho' + \frac{k^2}{R^2}}\,, \end{aligned} \tag{3.27}$$

and the solution is independent of θ, i.e., it is axisymmetric. For an unlimited shallow circular cylindrical shell* (of radius R' with generators x_2 = const.)

$$b_1 = 0\,, \qquad b_2 = -\frac{1}{R'} \tag{3.28}$$

and by a known Laplace transform [Erdélyi (1954)], (3.23) is

$$\left\{ \begin{matrix} I_1 \\ I_2 \end{matrix} \right\} = \pi \ell^{\mp i\tau'} K_0(\mp i\tau'), \quad \tau' = \frac{k\kappa}{2R'}\,(t-t'), \tag{3.29a}$$

hence at r = 0

$$\begin{aligned} I_1 + I_2 &= \pi^2 \left[J_0(\tau') \sin \tau' - Y_0(\tau') \cos \tau' \right], \\ I_1 - I_2 &= \pi^2 \left[J_0(\tau') \cos \tau' + Y_0(\tau') \sin \tau' \right], \end{aligned} \tag{3.29b}$$

where K_0, J_0, and Y_0 are Bessel functions in the usual notation of Watson (1944). For an unlimited shallow hyperbolic parabaloidal shell with

$$b_1 = -\, b_2 = b > 0\,, \tag{3.30}$$

by a known Laplace transform [Erdélyi (1954)], (3.23) yields

$$\begin{aligned} I_1 = I_2 &= \frac{\pi^2}{2} \left[H_0(\tau'') - Y_0(\tau'') \right], \\ \tau'' &= k\kappa b(t-t'), \end{aligned} \tag{3.31}$$

where H_0 is Struve's function of first order as defined by Watson (1944).

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*The unlimited shallow circular cylindrical shell is actually an unlimited parabolic cylindrical panel, which in the neighborhood of the x_1- axis may be regarded as equivalent to a circular cylindrical shell with edge sufficiently far removed from the shallow region of interest.

The solution for the unlimited flat plate ($b_1 = b_2 = 0$) cannot be obtained as a limiting case of the foregoing solution for unlimited shallow shells, since for $b_1 = b_2 = 0$, by (3.21)

$$a_0 = \frac{1}{\rho^2}, \qquad am = 0, \qquad m \geq 0 , \tag{3.32}$$

and the integral in (3.22) is divergent. This result is due to the fact that the spectral representation (3.2) for ∇^2 is not valid for solutions of the plate equations. For an unlimited plate Green's functions can be obtained with the aid of the extended definition of ∇^2 [see (4.20)], but the Green's function for the bending solution will contain a lnr term and an arbitrary function of time. In this respect the Green's function for an unlimited flat plate differs from those for unlimited shallow shells where the displacements vanish at $r = \infty$ and no arbitrary function of time is present. Thus, a comparison between quasi-static Green's functions for unlimited plates and shallow shells is not possible, and the Green's function for the flat plate will not be derived here. For a finite plate, however, no such difficulties arise.

The time variation of $\mathcal{G}_3^M$ $(x,t;\ x',t')$ at the source point $x_\alpha = x'_\alpha$ ($r = 0$) is shown in Fig. 1 for the spherical, cylindrical, and hyperbolic parabaloidal shallow shells. These results are obtained by numerical evaluation of (3.16), (3.25), (3.29), and (3.31), for the values

$$\frac{h}{R} = \frac{h}{R'} = hb = \frac{1}{30}, \qquad \nu = 0.3 , \qquad H = 0 \tag{3.33}$$

where $H = 0$ has been chosen since $H/hk \ll 1$ in most problems of practical interest. For qualitative comparison with these quasi-static Green's functions, since a dynamic Green's function for shells is not available, Fig. 1 shows the corresponding time variation of the dynamic Green's function $\mathcal{G}_3^M$ $(x,t;x't')$ at $r = 0$ for an unlimited flat plate as obtained by Nordgren and Naghdi (1962) which may be written as

$$\left[\mathcal{G}_3^M\right]_{n=0} = \frac{(1+\nu)\alpha\kappa}{4\pi}\, e^{-\kappa\eta_M(t-t')}\left[\overline{E}i\ \kappa\eta_M(t-t') - \ell n\gamma\eta\kappa\ (t-t')\right], \tag{3.34}$$

$$\text{for}\quad \frac{12\kappa}{h}\left(\frac{E}{\rho}\right)^{\frac{1}{2}} \ll 1 , \qquad \overline{E}i\tau = Ci(i\tau) - \frac{\pi i}{2} - Si(i\tau).$$

As seen in Fig. 1, the time variation for the three shells is similar and possesses a logarithmic singularity at $t = t'$ and decays exponentially as

$t - t' \to \infty$. It should be noted that the quasi-static Green's functions fail to meet a zero initial condition, i.e.,

$$\lim_{t-t' \to 0+} \mathcal{G}_3^M(x,t;x',t') \neq 0 \;, \tag{3.35}$$

for general values+ of x and x'. However, in the actual solution formula (2.53) for w, $\mathcal{G}_3^M(x,t;x',t')\,\theta_M(x',t')$ is integrated with respect to t' and if $\theta_M(x',t')$ is a piecewise continuous function of t' with $\theta_M(x',t') = 0$ for $t' < 0$, then w will be continuous in t and vanish at $t = 0$. On the other hand, if $\theta_M(x',t')$ is an impulse (δ-function), then the singularity in $\mathcal{G}_3^M(x,t;x',t')$ at $t = t'$ gives an unrealistic result for w, indicating that the dynamic solution should be employed. In the dynamic solution for the flat plate, $\mathcal{G}_3^M(x,t;x,t') = 0$ at $t = t'$ as desired. Mathematically, it is clear that zero initial conditions cannot be imposed on the quasi-static Green's function since time operators (inertia terms) are not present in the elasto-static equations.

It should be noted that $\mathcal{G}_3^M(x,t;x',t')$ given by (3.13) may differ considerably from $\mathcal{G}_3^M(x,t;x',t')$, since by (2.2), $\eta_N \ll \eta_M$ when $H/hk \ll 1$. Thus, $\mathcal{G}_3^N$ will decay more slowly than $\mathcal{G}_3^M$ as $t - t' \to \infty$ and for the spherical and cylindrical shallow shells, by (3.13), (3.25), and (3.29b), a damped oscillatory time variation of significant amplitude may be expected. For the hyperbolic parabaloidal shell (3.30), by (3.13) and (3.31), $\mathcal{G}_3^N(x,t;x',t') = 0$ at $x = x'$.

3.2 A Class of Rectangular Shallow Shells. Let us consider the class of shallow shells (3.7) now with $0 \le x_\alpha \le a_\alpha$ and boundary conditions (2.4) and (2.32), i.e.,

$$\begin{aligned} &w = v_2 = 0 \,, \quad M_{11} = H_N^* \,, \quad N_{11} = N_N^* \,, \quad \text{on } x_1 = 0, a_1 \,, \\ &w = v_1 = 0 \,, \quad M_{22} = H_N^* \,, \quad N_{22} = N_N^* \,, \quad \text{on } x_2 = 0, a_2 \,, \end{aligned} \tag{3.36}$$

$$\theta_N = \theta_N^* \,, \qquad \theta_M = \theta_M^* \,, \qquad \text{on } x_\alpha = 0, a_\alpha \,. \tag{3.37}$$

By (2.19) and (3.9), the homogeneous form of (3.36) is met if

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+A similar situation occurs in the theory of elasticity where, e.g., the thermoelastic Green's functions for the entire space according to quasi-static theory as given by Parkus (19'9) do not vanish as $t - t' \to 0$.

$$w = \frac{\partial^2 w}{\partial x_1{}^2} = F = \frac{\partial^2 F}{\partial x_1{}^2} = 0\,, \qquad \text{on } x_1 = 0, a_1,$$
$$w = \frac{\partial^2 w}{\partial x^2} = F = \frac{\partial^2 F}{\partial x_2{}^2} = 0, \text{ on } x_2 = 0, a_2\,, \tag{3.38}$$

$$\theta_N = \theta_M = 0\,, \qquad x_\alpha = 0, a_\alpha. \tag{3.39}$$

In view of (3.38) the domain of the operators $L_\alpha = \frac{\partial^2}{\partial x_\alpha{}^2}$ appearing in (2.1) and (3.8) may be taken as the set of all integrable square functions which vanish at $x_\alpha = 0, a_\alpha$ and have piecewise continuous and integrable square second derivatives with respect to x_α. Since by (3.38) $L_\beta N$ vanishes at $x_\alpha = 0, a_\alpha$, the operators in (3.8) of the form $L_\alpha L_\beta N$ be interpreted as L_α (with the domain just specified) acting on $L_\beta N$ provided that N has piecewise continuous and integrable square fourth partial derivatives. Then, since L_1 and L_2 commute, we may again follow the method established by Friedman (1956) and regard L and $\frac{\partial}{\partial t}$ as constants in solving for the Green's function. The result is interpreted using for $\frac{\partial}{\partial t}$ the spectral representation (3.2) and for L_1 and L_2 the spectral representations

$$\delta(x_1 - x_1') = \frac{2}{a_1} \sum_{m=1}^{\infty} \sin \frac{m\pi x_1}{a_1} \quad \sin \frac{m\pi x_1'}{a_1}\,,$$
$$\delta(x_2 - x_2') = \frac{2}{a_2} \sum_{n=1}^{\infty} \sin \frac{n\pi x_2}{a_2} \quad \sin \frac{n\pi x_2'}{a_2}\,, \tag{3.40}$$

respectively, which are given by Friedman (1956, p. 249). In this manner, (3.1) yields the temperature Green's function

$$G_N(x,t;x',t') = \frac{4\kappa}{a_1 a_2} \quad e^{-\kappa\eta_N(t-t')} \sum_{\substack{m=1\\n=1}}^{\infty} \exp\left[-\kappa(m'^2 + n'^2)(t-t')\right.$$
$$\sin m'x_1 \sin m'x_1' \quad \sin n'x_2 \quad \sin n'x_2'\,, \tag{3.41}$$

where

$$m' = \frac{m\pi}{a_1}\,, \qquad n' = \frac{n\pi}{a_2}\,, \tag{3.42}$$

and a corresponding expression for $G_M(x,t;x',t')$ follows upon replacing η_N by η_M in (3.41). A similar technique applied to (3.11) results in

$$\mathscr{G}_3^N = \frac{2k\kappa\alpha}{a_1 a_2 i} \; e^{-\kappa\eta_N(t-t')} \; (I_1' - I_2') \; ,$$
$$F^N = \frac{2Eh\kappa\alpha}{a_1 a_2} \; e^{-\kappa\eta_N(t-t')} \; (I_1' + I_2') \; , \tag{3.43}$$

where

$$\begin{Bmatrix} I_1' \\ I_2' \end{Bmatrix} = \sum_{\substack{m=1 \\ n=1}}^{\infty} \frac{(m'^2 + n'^2) \exp\left\{-\kappa(m'^2+n'^2)(t-t')\right\}}{\left[(m'^2+n'^2)^2 \pm ik(b_2 m'^2 + b_1 n'^2)^2\right]} \tag{3.44}$$

$$\sin m'x_1 \quad \sin m'x_1' \quad \sin n'x_2 \quad \sin n'x_2' \quad .$$

By (3.10), (3.41), and (3.43) the remaining components $\mathscr{G}_2^N$ of Green's function $\mathscr{G}_i^N$ are

$$\mathscr{G}_1^N = \frac{4\kappa\alpha}{a_1 a_2} \; e^{-\kappa\eta_N(t-t')} \sum_{\substack{m=1 \\ n=1}}^{\infty} \frac{1}{m'} (\cos m'x_1 - 1)$$

$$\left\{\frac{(m'^2+n'^2)^3 \; (n'^2-\nu m'^2) + b_1 k^2 (m'^2+n'^2)(b_2 m'^2 + b_1 n'^2)}{(m'^2+n'^2)^4 + k^2(b_2 m'^2 + b_1 n'^2)} \right. \tag{3.45}$$

$$\left. - 1\right\} \exp\left\{-\kappa(m'^2+n'^2)(t-t') \right. \sin m'x_1' \quad \sin n'x_2 \quad \sin n'x_2' \; ,$$

and a similar expression for $\mathscr{G}_2^N$ $(x,t;x',t')$ which is given by (3.45), with M and N as well as subscripts 1 and 2 interchanged. By the same procedure we obtain the Green's functions

$$\mathscr{G}_3^M = \frac{2(1+\nu)\kappa\alpha}{a_1 a_2} \; e^{-\kappa\eta_M(t-t')} \; (I_1' + I_2') \; ,$$

$$\mathscr{G}_1^M = \frac{4(1+\nu)\kappa\alpha}{a_1 a_2} \; e^{-\kappa\eta_M(t-t')} \sum_{\substack{m=1 \\ n=1}}^{\infty} \frac{1}{m} (\cos m'x_1 - 1) \tag{3.46a}$$

$$\left\{\frac{(m'^2+n'^2)(b_2 m'^2 + b_1 n'^2)(n'^2-\nu m'^2) - b_1 (m'^2+n'^2)^3}{(m'^2+n'^2)^4 + k^2(b_2 m'^2 + b_1 n'^2)^2} \right. \tag{3.46b}$$

$$\left. - 1\right\} \exp\left\{-\kappa(m'^2+n'^2)(t-t') \right. \sin m'x_1' \quad \sin n'x_2 \quad \sin n'x_2 \; ,$$

a similar expression for $\mathscr{G}_2^M$ $(x,t;x',t')$, and

$$F^M = \frac{-2(1+\nu)Eh\kappa\alpha}{a_1 a_2 k i} \; e^{-\kappa\eta_M(t-t')} \; (I_1' - I_2') \; , \tag{3.46c}$$

where I_1' and I_2' are defined by (3.44).

As an application of the formulas of Section 2 and the Green's functions just obtained for a class of rectangular shallow shells, we consider the edge conditions (3.37) with

$$\begin{aligned} \theta_N^*(x,t) &= f(x_1), \qquad x_2 = 0, \quad t \geq 0, \\ &= -f(x_1), \qquad x_2 = a_2, \quad t \geq 0, \\ &= 0, \qquad x_1 = 0, a_1, \\ &= 0, \qquad t < 0, \end{aligned} \tag{3.47a}$$

and also take

$$H_N^* = N_N^* = Q_N = Q_M = \theta_N^\circ = \theta_M^\circ = 0, \tag{3.47b}$$

where $f(x_1)$ is a continuous function of integrable square on $0 \leq x_1 \leq a_1$. By (2.16), (2.53), and (3.47), we have

$$\begin{aligned} \theta_N(x,t) &= \int_0^t dt' \int_0^{a_1} \left\{ \left[\frac{\partial}{\partial x_2'} G_N(x,t;x',t') \right]_{x_2=0} \right. \\ &\quad \left. + \left[\frac{\partial}{\partial x_2'} G_N(x,t;x'\ t') \right]_{x_2=a_2} \right\} f(x_1')\, dx_1', \\ U_1(x,t) &= \int_0^t \int_0^a \left\{ \left[\frac{\partial}{\partial x_2'} \mathcal{G}_1^N (x,t;x',t') \right]_{x_2=0} \right. \\ &\quad \left. + \left[\frac{\partial}{\partial x_2'} \mathcal{G}_1^N (x,t;x',t') \right]_{x_2=a_2} \right\} f(x_1') dx_1' \end{aligned} \tag{3.48}$$

which with the Green's functions (3.41) to (3.45) become

$$\theta_N(x,t) = \frac{4}{a_2} \sum_{\substack{m=1 \\ n=1}}^{\infty} \frac{1 - \exp\left\{-\kappa(m'^2 + 4n'^2 + \eta_N)t\right\}}{m'^2 + 4n'^2 + \eta_N} \tag{3.49a}$$

$$2n' fm \quad \sin m'x_1 \quad \sin 2n'x_2 ,$$

$$\begin{aligned} U_1(x,t_1) &= \frac{4\alpha}{a_2} \sum_{\substack{m=1 \\ n=1}}^{\infty} \frac{1 - \exp\left\{-\kappa(m'^2 + 4n'^2 + \eta_N)t\right\}}{m'^2 + 4n'^2 + \eta_N} \\ &\quad \left\{ \frac{(m'^2 + 4n'^2)^3 (4n'^2 - \nu m'^2) + b_1 k^2 (m'^2 + 4n'^2)(b_2 m'^2 + b_1 4n'^2)}{(m'^2 + 4n'^2)^4 + k^2 (b_2 m'^2 + b_1 4n'^2)^2} \right. \\ &\quad \left. - 1 \right\} \frac{2n'}{m'} (\cos m'x_1 - 1)\ fm \quad \sin 2n'x_2, \end{aligned} \tag{3.49b}$$

a similar expression for $U_2(x,t)$ and

$$U_3(x,t) = -\frac{4k^2\alpha}{a_2}\sum_{\substack{m=1\\n=1}}^{\infty} \frac{1-\exp\left\{-\kappa\,(m'^2+4n'^2+\eta_N)t\right\}}{m'^2+4n'^2+\eta_N}$$

$$\left\{(m'^2+4n'^2)^4 + k^2\,(b_2m'^2+b_14n'^2)^2\right\}^{-1} \tag{3.49c}$$

$$(m'^2+4n'^2)\,(b_2m'^2+b_14n'^2)\,2n'fm \quad \sin m'x_1 \ \sin 2n'x_2 \ ,$$

where

$$fm = \frac{2}{a_1}\int_0^{a_1} f(x_1) \ \sin m'x_1 \ dx_1 \ . \tag{3.50}$$

By (2.57), (3.9), (3.43) and (3.47) the stress resultants are found to be

$$\left\{\begin{matrix} N_{11}(x,t) \\ N_{22}(x,t) \\ N_{12}(x,t)\end{matrix}\right\} = -\frac{4Eh\alpha}{a_2}\sum_{\substack{m=1\\n=1}}^{\infty} \frac{1-\exp\left\{-\kappa(m'^2+4n'^2+\eta_M)t\right\}}{m'^2+4n'^2+\eta_N}$$

$$\frac{(m'^2+4n'^2)^3\ fm}{(m'^2+4n'^2)^4 + k^2\,(b_2m'^2+b_14n'^2)^2} \tag{3.51}$$

$$\left\{\begin{matrix} -(2n')^3 \ \sin m'x_1 \ \sin 2nx_2 \\ m'^2 2n' \ \sin m'x_1 \ \sin 2n'x_2 \\ m'\,4n'^2 \cos m'x_1 \ \cos 2n'x_2 \end{matrix}\right\} ,$$

and similar expressions for $M_{\alpha\beta}$ and Q_α are easily obtained. From the theory of Fourier series it may be shown in the usual manner [see, e.g., Friedman (1956), p.271] that θ_N (3.49a) converges nonuniformly to the boundary values (3.47a). To this end we write (3.49a) as

$$\theta_N = 2\sum_{\substack{m=1\\n=1}}^{\infty} \frac{1}{n\pi}\ fm \quad \sin m'x_1 \quad \sin 2n'x_2$$

$$+ \sum_{\substack{m=1\\n=1}}^{\infty} A\,(m,n) \quad \sin m'x_1 \quad \sin 2n'x_2 \ , \tag{3.52}$$

where $A\,(m,n) = \theta\,(\frac{1}{n^2})$ as $n \to \infty$. Thus, the second series converges uniformly and is zero at $x_2 = 0, a_2$, while the first series converges to

$$(1 - \frac{2x_2}{a_2}) \ \sin m'x_1 \ fm = (1 - \frac{2x_2}{a_2}) \ f(x_1) \ , \ 0 < x_2 < a_2 \tag{3.53}$$

which approaches $f(x_1)$ and $-f(x_1)$ as x_2 approaches 0 and a_2, respectively. In a similar manner the remaining homogeneous boundary conditions (3.36) and (3.47) may be shown to be satisfied by (3.49) and (3.51) with the series

converging uniformly in each case. Note, however, that on $x_2 = 0$ and $x_2 = a_2$, N_{11} converges nonuniformly to the values $-Eh\,\alpha\, f(x_1)$ and $Eh\alpha f(x_1)$, respectively.

4. IMPROVED THEORY OF SHALLOW SHELLS. GREEN'S FUNCTIONS FOR AN UNLIMITED SHALLOW SPHERICAL SHELL.

4.1 Basic Equations. Formulation of Elastostatic Problems. We recall from the note by Naghdi (1956) that the effect of transverse shear deformation may be included in the theory of shallow shells [in a manner similar to Reissner's (1945) improved theory of flat plates] by replacing (2.18b) and (2.21) of the classical theory with

$$2\kappa_{\alpha\beta} = \nabla_\alpha \beta_\beta + \nabla_\beta \beta_\alpha \,, \tag{4.1}$$

and

$$Q^\alpha = \frac{5Eh}{12(1+\nu)c}\, a^{\alpha\beta}\,(B_\beta + \nabla_\beta w) \,, \tag{4.2}$$

respectively. The components of β_α represent the changes in slope of the normal to the shell middle surface, and the tracer $c = 1$ is introduced in (4.2) to isolate the effect of transverse shear deformation in future manipulations. Also, in some equations it will be possible to let $c = 0$ and obtain the corresponding equation for classical theory.

In the absence of tangential surface tractions ($p^\alpha = 0$), (2.17a) leads in the usual manner to

$$N^{\alpha\beta} = \epsilon^{\alpha\lambda}\, \epsilon^{\beta\eta}\, \nabla_\lambda \nabla_\eta\, F \,, \tag{4.3}$$

where, by the compatability equation associated with (2.18a), together with (2.19a) and (4.3), the Airy stress function F must satisfy

$$\nabla^2\nabla^2\, F + Eh\epsilon^{\alpha\lambda}\, \epsilon^{\beta\eta}\, \nabla_\alpha\nabla_\beta\, z\, \nabla_\lambda\nabla_\eta w + Eh\, \alpha\, \nabla^2 \theta_N = 0 \tag{4.4}$$

just as in classical theory. Introduction of (2.19b), (4.1), (4.2), and (4.3) into (2.17b,c) results in

$$\frac{5Eh}{12(1+\nu)c}\,(\nabla^\alpha\, \beta_\alpha + \nabla^2\, w) + \epsilon^{\alpha\lambda}\, \epsilon^{\beta\eta}\, \nabla_\alpha\nabla_\beta\, z\, \nabla_\lambda\nabla_\eta\, F + p = 0, \tag{4.5}$$

$$\frac{5Eh}{12(1+\nu)c}\ (\beta_\alpha + \nabla_\alpha w) - D\ A^{\alpha\beta\lambda\eta}\ \nabla_\eta \nabla_\beta \beta_\lambda$$
$$+ D\,\alpha\,\nabla^2\ \theta_M = 0. \qquad (4.6)$$

which contain the effect of transverse shear deformation. Next, by the two-dimensional version of the Stokes-Helmholtz theorem of vector analysis we may write

$$\beta_\alpha = \nabla_\alpha\ H' + a_{\alpha\eta}\epsilon^{\eta\lambda}\ \nabla_\lambda K'\ , \qquad (4.7)$$

and then (4.6) becomes

$$\nabla^2 H' - \lambda^{-1}\ (H' + w) - (1+\nu)\ \alpha\theta_M = f,$$
$$\nabla^2 K' - \frac{10}{h^2 c}\ K' = \frac{2}{1-\nu}\ g\ , \qquad (4.8)$$

where

$$\nabla_\alpha f + a_{\alpha\eta}\epsilon^{\eta\lambda}\ \nabla_\lambda g = 0, \qquad \nabla^2 f = \nabla^2 g = 0,$$
$$\lambda = \frac{ch^2}{5(1-\nu)}\ . \qquad (4.9)$$

Letting

$$H = H' + \lambda f\ , \qquad K = K' + \lambda g, \qquad (4.10)$$

by (4.9), equations (4.7) and (4.8) read

$$\beta_\alpha = \nabla_\alpha H + a_{\alpha\eta}\epsilon^{\eta\lambda}\ \nabla_\lambda K, \qquad (4.11)$$

$$\nabla^2 H - \lambda^{-1}(H+w) - (1+\nu)\alpha\theta_M = 0, \qquad (4.12)$$

$$\nabla^2 K - \frac{10}{h^2 c}\ K = 0 \qquad (4.13)$$

which are the same for all shallow shells, and (4.5) is

$$\lambda D(\nabla^2 H + \nabla^2 w) + \epsilon^{\alpha\lambda}\ \epsilon^{\beta\eta}\ \nabla_\alpha \nabla_\beta\ z\ \nabla_\lambda \nabla_\eta\ F + p = 0. \qquad (4.14)$$

Thus, all solutions in the improved theory of shallow shells may be represented in terms of F, w, H, and K which are governed by (4.4), (4.12), (4.13), and (4.14). It may be noted that for flat plates ($z = 0$), (4.12) and (4.13) remain unchanged while (4.4) reduces to an equation for F alone and (4.14)

reduces to an equation containing H and w. For bending of flat plates the system (4.12), (4.13), and (4.14) with $z = 0$ is equivalent to the formulation by Reissner (1945) which involves a stress function representation for Q^{α}.

4.2 Green's Functions for an Unlimited Shallow Spherical Shell Under Normal Traction and Temperature Field. For a shallow spherical shell (of radius R) referred to rectangular Cartesian coordinates x_{α}

$$z = \frac{-x_{\alpha}x_{\alpha}}{2R} \quad , \quad \nabla_{\alpha}\nabla_{\beta}\, z = \frac{-\delta_{\alpha\beta}}{R} , \tag{4.15}$$

the basic equations (4.4) and (4.14) simplify somewhat, while (4.11) and (4.12) remain unchanged. For an unlimited shallow spherical shell the solution may be written in the integral form

$$w(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\Big[w_p(x;x')\ p(x') + w_M(x;x')\ \theta_M\ (x') + w_N(x;x')\ \theta_N(x')\Big]\ dx_1'\ dx_2' \ , \tag{4.16}$$

and similar expressions for F, H, and K, where the three sets of Green's functions wp, Fp, Hp, Kp; w_M, F_M, H_M, K_M; and w_N, F_N, H_N, K_N are solutions of the basic equations with $p = \delta(x-x')$, $\theta_M = \theta_N = 0$; $\theta_M = \delta(x-x')$, $p = \theta_N = 0$; and $\theta_N = \delta(x-x')$, $p = \theta_M = 0$, respectively. Thus, all three sets of Green's functions may be obtained from solution of

$$\nabla^2\nabla^2 F_G - \frac{Eh}{R}\nabla^2 w_G = -\ Eh\alpha\overline{\theta}_N\ \delta(x-x') \ , \tag{4.17a}$$

$$\lambda(\nabla^2 H_G + \nabla^2 w_G) - \frac{1}{RD}\nabla^2 F_G = -\ \frac{1}{D}\ \overline{p}\ \delta(x-x'), \tag{4.17b}$$

$$\nabla^2 H_G - \lambda^{-1}\ (H_G + w_G) = (1 + \nu)\ \alpha\overline{\theta}_M\ \delta(x-x') \ , \tag{4.17c}$$

$$\nabla^2 K_G - \frac{10}{h^2c}\ K_G = 0 \ , \tag{4.17d}$$

where $\overline{p}$, $\overline{\theta}_M$, and $\overline{\theta}_N$ are constants taken in turn as $\overline{p} = 1$, $\overline{\theta}_M = \overline{\theta}_N = 0$; $\overline{\theta}_M = 1$, $\overline{p} = \overline{\theta}_N = 0$; and $\overline{\theta}_N = 1$, $\overline{p} = \overline{\theta}_M = 0$, for each set of Green's functions.

To solve (4.17) we introduce polar coordinates r and θ through

$$x_1 - x_1' = r\cos\theta, \qquad x_2 - x_2' = r\sin\theta, \tag{4.18}$$

and recall that

$$\delta(x-x') = \frac{1}{2\pi r}\,\delta(r) \;, \tag{4.19}$$

and the definition of the Laplacian may be extended to functions U(r) not regular at r = 0 according to

$$\nabla^2 U = \frac{1}{r}\frac{\partial_o}{\partial r}\left(r\,\frac{\partial_o U}{\partial r}\right) + \frac{\delta'(r)}{r}\lim_{r\to 0} rU + \frac{\delta(r)}{r}\lim_{r\to 0} r\,\frac{\partial_o U}{\partial r} \;, \tag{4.20**}$$

where $\partial_o/\partial r$ indicates differentiation in the ordinary (not symbolic) sense. With the aid of (4.19) and (4.20), (4.17) may be written as

$$\begin{aligned}
&\left[\nabla^2\nabla^2 F_G - \frac{Eh}{R}\nabla^2 w_G\right] + \frac{\delta(r)}{r}\left[\lim_{r\to 0} r\,\frac{\partial}{\partial r}\left(\nabla^2 F_G - \frac{Eh}{R}w_G\right)\right] \\
&+ \frac{\delta'(r)}{r}\left[\lim_{r\to 0} r\left(\nabla^2 F_G - \frac{Eh}{R}w_G\right)\right] + \nabla^2\,\frac{\delta(r)}{r}\left[\lim_{r\to 0} r\,\frac{\partial_o F_G}{\partial r}\right. \\
&\left. + \frac{Eh\alpha}{2\pi}\,\overline{\theta}_N\right] + \nabla^2\,\frac{\delta'(r)}{r}\left[\lim_{r\to 0} r\,F_G\right] = 0 \;,
\end{aligned} \tag{421a}$$

$$\begin{aligned}
&\left[\lambda^{-1}(\nabla^2 H_G + \nabla^2 w_G) - \frac{1}{RD}\nabla^2 F_G\right] + \frac{\delta(r)}{r}\left[\lim_{r\to 0}\frac{\partial_o}{\partial r}\left(\lambda^{-1}(H_G + w_G)\right.\right. \\
&\left.\left. - \frac{1}{RD}F_G\right) + \frac{\overline{p}}{2\pi D}\right] + \frac{\delta'(r)}{r}\left[\lim_{r\to 0}\left(\lambda^{-1}(H_G + w_G) - \frac{1}{RD}F_G\right)\right] = 0,
\end{aligned} \tag{4.21b}$$

$$\begin{aligned}
&\left[\nabla^2 H_G - \lambda^{-1}(H_G + w_G)\right] + \frac{\delta(r)}{r}\left[\lim_{r\to 0} r\,\frac{\partial_o H_G}{\partial r}\right. \\
&\left. - \frac{(1+\nu)\alpha}{2\pi}\,\overline{\theta}_M\right] + \frac{\delta'(r)}{r}\left[\lim_{r\to 0} r\,H_G\right] = 0 \;,
\end{aligned} \tag{4.21c}$$

$$\begin{aligned}
&\left[\nabla^2 K_G - \frac{10}{h^2 c}K_G\right] + \frac{\delta(r)}{r}\left[\lim_{r\to 0} r\,\frac{\partial_o K_G}{\partial M}\right] \\
&+ \frac{\delta'(r)}{r}\left[\lim_{r\to 0} r\,K_G\right] = 0
\end{aligned} \tag{4.21d}$$

which are satisfied if each expression in brackets vanishes. The first brackets in (4.21a,b,c,d) lead to the general symmetric solution

** The corresponding extended definition of the three-dimensional Laplacian is derived by Friedman (1959), p. 255.

$$w_G = - B_1 - B_2 \ln kr + (\lambda\zeta_1^2 - 1)\Big[B_3 I_o(\zeta_1 r)$$

$$+ B_4 K_o(\zeta_1 r)\Big] + (\lambda\zeta_2^2 - 1)\Big[B_5 I_o(\zeta_2 r) + B_6 K_o(\zeta_2 r)\Big] , \tag{4.22a}$$

$$H_G = B_1 + B_2 \ln kr + B_3 I_o(\zeta_1 r) + B_4 K_o(\zeta_1 r)$$

$$+ B_5 I_o(\zeta_2 r) + B_6 K_o(\zeta_2 r) , \tag{4.22b}$$

$$K_G = B_7 I_o\left[\frac{r}{h}\left(\frac{10}{c}\right)^{\frac{1}{2}}\right] + B_8 K_o\left[\frac{r}{h}\left(\frac{10}{c}\right)^{\frac{1}{2}}\right], \tag{4.22c}$$

$$F_G = B_9 + B_{10} \ln kr + RD\Big[B_3 \zeta_1^2 I_o(\zeta_1 r) + B_4\zeta_1^2 K_o(\zeta_1 r) + B_5\zeta_2^2 I_o(\zeta_2 r) + B_6 K_o(\zeta_2 r)\Big] , \tag{4.22d}$$

where $B_1, B_2, \ldots B$ are constants and

$$\left\{\begin{matrix}\zeta_1\\ \zeta_2\end{matrix}\right\} = k\left[\frac{k^2\lambda}{2} \pm i\left(1 - \frac{k^4\lambda^2}{4}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}} ,$$

$$\mathrm{Re}\ \zeta_1, \zeta_2 > 0 , \quad k^4 = \frac{12(1-\nu^2)}{h^2 R^2} , \quad \frac{k^2\lambda}{2} = \frac{c}{5}\left[\frac{3(1+\nu)}{1-\nu}\right]^{\frac{1}{2}} \quad \frac{h}{R} < 1 . \tag{4.23}$$

Recalling that

$$I_o(\zeta r) \to \frac{e^{\zeta r}}{\sqrt{2\, r\zeta}} \to \infty , \quad \text{as} \quad r \to \infty , \qquad \mathrm{Re}\ \zeta > 0 , \tag{4.24}$$

then in order that w_G and K_G have at most a singularity of $O(\ln r)$ as $r \to \infty$, $B_3 = B_5 = B_7 = 0$, and without loss in generality, $B_1 = B_9 = 0$. Further, since

$$\lim_{r\to 0} r\, K_o(\zeta r) = \lim_{r\to 0} \nabla^2 K_o(\zeta r) = 0,$$

$$\lim_{r\to 0} r \frac{\partial_o}{\partial r} K_o(\zeta r) = -1 , \quad \lim_{r\to 0} r \frac{\partial_o}{\partial r}\nabla^2 K_o(\zeta r) = -\zeta^2 , \tag{4.25}$$

(4.22) meet (4.21), provided

$$B_4 = \frac{1}{2\pi(\zeta_1^2 - \zeta_2^2)}\left[\frac{\bar{p}}{D} + Rk^4\alpha\theta_N + \zeta_2^2(1+\nu)\,\alpha\bar{\theta}_M\right] ,$$

$$B_6 = \frac{-1}{2\pi(\zeta_1^2 - \zeta_2^2)}\left[\frac{\bar{p}}{D} + Rk^4\alpha\bar{\theta}_N + \zeta_1^2(1+\nu)\,\alpha\bar{\theta}_M\right] \tag{4.26}$$

$$B_2 = B_8 = 0 , \qquad B_{10} = \frac{\bar{p}R}{2\pi} .$$

Thus, the solution (4.22) which determines the Green's functions is

$$w_G = \frac{(\lambda\zeta_1^2 - 1)}{2\pi(\zeta_1^2 - \zeta_2^2)}\left[\frac{\bar{p}}{D} + Rk^4\alpha\bar{\theta}_N + \zeta_2^2(1+\nu)\alpha\bar{\theta}_M\right]K_o(\zeta_1 r)$$

$$-\frac{(\lambda\zeta_2^2 - 1)}{2\pi(\zeta_1^2 - \zeta_2^2)}\left[\frac{\bar{p}}{D} + Rk^4\alpha\bar{\theta}_N + \zeta_1^2(1+\nu)\alpha\bar{\theta}_M\right]K_o(\zeta_2 r),$$

$$F_G = \frac{\bar{p}R}{2\pi}\ln kr + \frac{\zeta_1^2}{2\pi(\zeta_1^2 - \zeta_2^2)}\left[\bar{p}R + Eh\,\alpha\bar{\theta}_N\right. \tag{4.27}$$

$$\left.+ RD\zeta_2^2(1+\nu)\alpha\bar{\theta}_M\right]K_o(\zeta_1 r) - \frac{\zeta_2^2}{2\pi(\zeta_1^2 - \zeta_2^2)}\left[\bar{p}R + Eh\,\alpha\theta_N + RD\zeta_1^2(1+\nu)\alpha\theta_M\right]K_o(\zeta_2 r),$$

$$H_G = \frac{1}{2\pi(\zeta_1^2 - \zeta_2^2)}\left[\left(\frac{\bar{p}}{D} + RK^4\alpha\bar{\theta}_N + \zeta_2^2(1+\nu)\alpha\theta_M\right)K_o(\zeta_1 r)\right.$$

$$\left.-\left(\frac{p}{D} + Rk^4\alpha\bar{\theta}_N + \zeta_1^2(1+\nu)\alpha\bar{\theta}_M\right)K_o(\zeta_2 r)\right].$$

To obtain corresponding results for the classical theory of shallow shells we neglect the effect of transverse shear deformation by setting $c = 0$, in which case for $\bar{\theta}_M = \bar{\theta}_N = 0$ (4.27) reduces to the solution given by Reissner (1946) for an unlimited shallow spherical shell under concentrated load, and for $\bar{p} = 0$ (4.27) reduces to the solutions given by Flügge and Conrad (1956) for concentrated temperature resultants.

With the aid of (4.25), in the neighborhood of $r = 0$ (4.27) is

$$w_G = \frac{1}{8\pi}\left(\frac{\bar{p}}{D} + Rk^4\alpha\,\bar{\theta}_N\right)\left[(1-\lambda^2k^4)\,r^2\ln kr - 4\ln kr + \ldots\right]$$

$$-\frac{(1+\nu)\alpha}{2\pi}\bar{\theta}_M\left[\ln kr - \frac{1}{4}\lambda^2k^4r^2\ln kr + \ldots\right],$$

$$F_G = -\frac{R\lambda k^4}{8\pi}\bar{p}\left[r^2\ln kr + \ldots\right] - \frac{Eh\,\alpha\bar{\theta}_N}{2\pi}\left[\ln kr\right. \tag{4.28}$$

$$\left.+\frac{1}{4}\lambda k^2r^2\ln kr + \ldots\right] + \frac{(1+\nu)\alpha RDk^2}{8\pi}\bar{\theta}_N\left[r^2\ln kr + \ldots\right],$$

$$H_G = -\frac{1}{8\pi D}\left(\frac{\bar{p}}{D} + Rk^4\alpha\bar{\theta}_M\right)\left[r^2\ln kr + \ldots\right] + \frac{(1+\nu)\alpha}{2\pi}\bar{\theta}_M\left[\ln kr + \ldots\right]$$

which differs from the classical theory ($c = \lambda = 0$). In particular, for the case $\bar{\theta}_M = \bar{\theta}_N = 0$, according to the improved theory $w_p \to \infty$ as $r \to 0$, while in the classical theory $w_p \to 0$ as $r \to 0$. In this connection we recall that in most other derivations of singular solutions for concentrated loads in the classical theory of plates and shells, the requirement $w_p \to 0$ as $r \to 0$ is imposed a priori; however, no such requirement is needed in the

foregoing treatment. Indeed, the requirement $w_p \to 0$ as $r \to 0$ for the improved theory would lead to an incorrect result for the Green's function as seen from (4.28). It may be recalled that Green's functions also are unbounded at the source point in the three-dimensional theory of elasticity.

To see that the $\overline{p}$ term in (4.27) represents a concentrated normal force at $r = 0$, by equilibrium considerations the total resultant normal force V_N^G, on a vanishingly small circular region with center at $r = 0$ is

$$V_N^G = - \lim_{r \to 0} \int_0^{2\pi} (Q_r^G + \frac{r}{R} N_r^G)\ r\ d\ \theta. \tag{4.29}$$

By (4.2), (4.3), and (4.11) in polar coordinates with polar symmetry, we have

$$Q_r^G = \frac{D}{\lambda} \frac{\partial}{\partial r} (H_G + w_G) \ , \quad N_r^G = \frac{\partial F_G}{\partial r} \ , \tag{4.30}$$

and by (4.28), (4.29) yields

$$V_N^G = \overline{p} \ , \tag{4.31}$$

as desired.

Green's functions for a flat plate with the effect of transverse shear deformation included cannot be obtained as a limiting case of (4.27) as $R \to \infty$. Instead, we must return to (4.21), let $\frac{1}{R} = 0$, and then in the same manner as for the shallow spherical shell we obtain

$$\begin{aligned} w_G &= \frac{\overline{p}}{8\pi D} \left[r^2 \ln \frac{r}{h} - 4\lambda\ (1 + \ln \frac{r}{h}) \right] - \frac{(1+\nu)\ \alpha}{2\pi}\ \overline{\theta}_M \ln \frac{r}{h} \ , \\ H_G &= - \frac{\overline{p}}{8\pi D}\ r^2 \ln \frac{r}{h} + \frac{(1+\nu)\ \alpha}{2\pi}\ \theta_M \ln \frac{r}{h} \ , \\ F_G &= - \frac{Eh\ \alpha}{2\pi}\ \overline{\theta}_N \ln \frac{r}{h} \ . \end{aligned} \tag{4.32}$$

When the effect of transverse shear deformation is neglected ($c = \lambda = 0$), (4.32) reduces to known results for classical plate theory. For the flat plate the improved and classical theories give different fundamental singularities only for the Green's function w_p .

5. APPENDIX:

EVALUATION OF CERTAIN DEFINITE INTEGRALS

Here we shall evaluate the definite integrals (3.20b) by contour integration around the unit circle in the complex z plane* where

$$z = r\,\ell^{2i\phi} . \tag{5.1}$$

Thus, on the unit circle C (r = 1)

$$\sin\phi = \frac{z^{\frac{1}{2}} - z^{-\frac{1}{2}}}{2i} , \qquad \cos\phi = \frac{z^{\frac{1}{2}} + z^{-\frac{1}{2}}}{2} ,$$

$$\cos 2m\phi = \frac{z^{m} + z^{-m}}{2} , \qquad dz = 2iz\,d\phi ,$$

and for $b_1 \neq b_2$ (3.20b) becomes

$$a_0(\pm k, \rho^2) = \frac{2}{\pi i(\lambda_2^2 - \lambda_1^2)} \oint_C \frac{dz}{(z-\beta)(z-\beta^{-1})} , \tag{5.2}$$

$$a_m(\pm k, \rho^2) = \frac{2}{\pi i(\lambda_2^2 - \lambda_1^2)} \oint_C \frac{(z^m + z^{-m})\,dz}{(z-\beta)(z-\beta^{-1})} , \tag{5.3}$$

where λ_1, λ_2, β are defined in (3.21b). Consider

$$|\beta|^2 = \beta\bar{\beta} = \frac{|\lambda_1|^2 + |\lambda_2|^2 - \lambda_1\bar{\lambda}_2 - \bar{\lambda}_1\lambda_2}{|\lambda_1|^2 + |\lambda_2|^2 + \lambda_1\bar{\lambda}_2 + \bar{\lambda}_1\lambda_2} , \tag{5.4}$$

where $\bar{\beta}$ denotes the complex conjugate of β. We may write λ_1 and λ_2 in the polar form

$$\lambda_1(\pm k, \rho^2) = R_1\,\ell^{\pm i\psi_1} , \quad \lambda_2(\pm k, \rho^2) = R_2\,\ell^{\pm i\psi_2} ,$$
$$-\frac{\pi}{4} \leq \psi_1, \psi_2 \leq \frac{\pi}{4} , \tag{5.5}$$

whence

$$\lambda_1\bar{\lambda}_2 + \bar{\lambda}_1\lambda_2 = 2\,R_1R_2 \cos(\psi_1 - \psi_2) ,$$
$$-\frac{\pi}{2} \leq (\psi_1 - \psi_2) \leq \frac{\pi}{2} . \tag{5.6}$$

Therefore, by (5.4) to (5.6)

$$\lambda_1\bar{\lambda}_2 + \bar{\lambda}_1\lambda_2 \geq 0 , \qquad |\beta| \leq 1 , \tag{5.7}$$

---

* This method is used, e.g., by Churchill (1948).

where the equality signs apply only when $\rho = 0$ for a special subclass of (3.7).

Excluding for now the case $\rho = 0$, the integrand in (5.2) has a pole inside the unit circle only at $z = \beta$, hence by the residue theorem we have

$$a_0 (\pm k, \rho^2) = \frac{1}{\lambda_1 \lambda_2} \quad . \tag{5.8}$$

Similarly, with the aid of the expansion

$$\frac{1}{(z - \beta)(z - \beta^{-1})} = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \beta^{j-2k} z^j , \tag{5.9}$$

(5.3) becomes

$$a_m (\pm k, \rho^2) = \frac{\beta^m + \beta^{-m}}{\lambda_1 \lambda_2} - \frac{4}{(\lambda_1^2 - \lambda_2^2)} \sum_{k=0}^{m-1} \beta^{m-2k-1} \quad ,$$

$$m \geqq 1 \quad , \qquad b_1 \neq b_2 \ . \tag{5.10}$$

For the special case $\rho = 0$, since $a_m (\pm k, \rho^2)$ appears in the integrand of (3.2), it is sufficient to take

$$a_m (\pm k, 0) = \lim_{\rho \to 0} (\pm k, \rho^2) \ , \tag{5.11}$$

where, by (5.9) and (5.10) the limit in (5.11) exists for $b_1 \neq b_2$. For the spherical shell $b_1 = b_2$, by (3.21b), $\lambda_1 = \lambda_2$ and (3.20b) yields (5.8) and

$$a_m (\pm k, \rho^2) = 0, \qquad m \geqq 1. \tag{5.12}$$

Thus we have obtained the results expressed by (3.21a).

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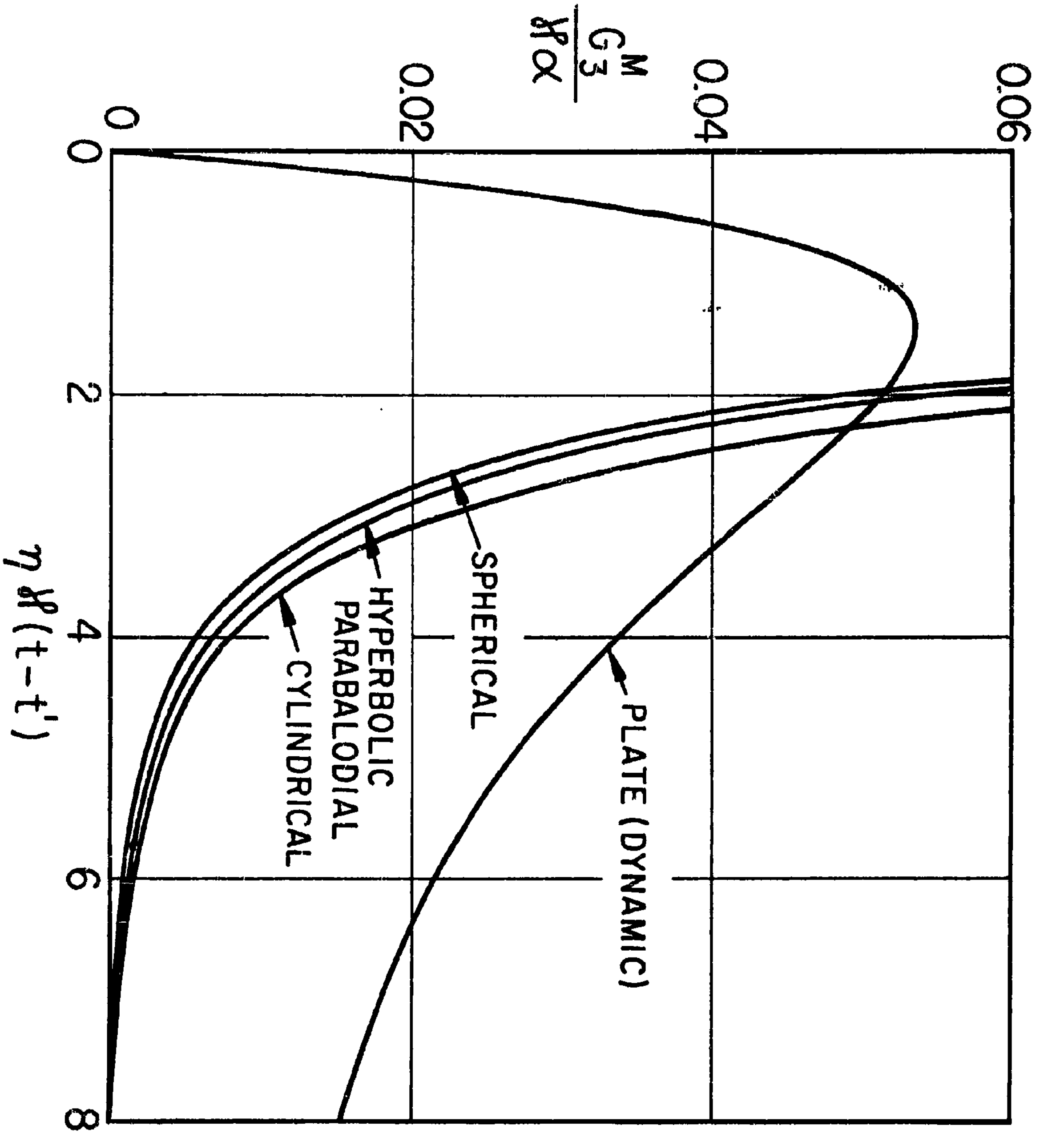

Figure 1. Time variation of the Green's function $G_3^M(x;t;x';t')$ at $x_\alpha = x'_\alpha$ ($r = 0$) according to quasi-static theory for unlimited spherical, cylindrical, and hyperbolic parabaloidal shallow shells ($\frac{h}{R} = \frac{h}{R'} = hb = \frac{1}{30}$, $\nu = 0.3$) and for qualitative comparison according to dynamic theory ($\frac{12\kappa}{h} (\frac{E}{\rho})^{\frac{1}{2}} \ll 1$) for an infinite flat plate.

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